3. Quantum–Classical Correspondence for the Electromagnetic Field I: The Glauber–Sudarshan P Representation

In Chap. 1 we developed a formalism to handle dissipative problems in quantum mechanics. The central result of this formalism was the operator master equation for the reduced density operator ρ of a dissipative system. This equation can be written formally as

$$\dot{\rho} = \mathcal{L}\rho, \tag{3.1}$$

where $\mathcal{L}$ is a generalized Liouvillian, or "superoperator", which acts, not on the states, but on the operators of the system. In a specific application $\mathcal{L}$ is defined by an explicit expression in terms of various commutators involving system operators. While it is generally not possible to solve the operator master equation directly to find $\rho(t)$ in operator form, we have seen that alternative methods of analysis are available to us. We can derive equations of motion for expectation values, and if these form a suitable closed set, solve these equations for time-dependent operator averages. Alternatively, we may choose a representation and take matrix elements of (3.1) to obtain equations of motion for the matrix elements of ρ. We have also seen how equations of motion for one-time operator averages can be used to obtain equations of motion for two-time averages (correlation functions) using the quantum regression formula.

We are now going to meet an entirely new approach to the problem of solving the operator master equation and calculating operator averages and correlation functions. For the present we will only consider the electromagnetic field —i.e. the harmonic oscillator. In Chap. 6 we will generalize the techniques learned here to collections of two-level atoms. This new approach establishes a correspondence between quantum-mechanical operators and ordinary (classical) functions, such that quantities of interest in a quantum-mechanical problem can be calculated using the methods of classical statistical physics. Under this correspondence the operator master equation transforms into a partial differential equation for a quasidistribution function which corresponds to (represents) ρ. For the damped harmonic oscillator this quasidistribution function is a function of the classical phase-space variables q and p, or alternatively, the complex variables $\alpha = (m\omega q + ip)/\sqrt{2\hbar m\omega}$ and $\alpha^* = (m\omega q - ip)/\sqrt{2\hbar m\omega}$ that correspond to the operators a and $a^\dagger$. Operator averages, written in an appropriate order (e.g. normal order), are

calculated by integrating functions of these classical variables against the quasidistribution function, in the same manner in which we take classical phase-space averages. This *quantum–classical correspondence* is particularly appealing when the partial differential equation corresponding to the operator master equation is a Fokker–Planck equation. Fokker–Planck equations are familiar from classical statistical physics, and in this context they have been studied extensively [3.1]. When the operator master equation becomes a Fokker–Planck equation, analogies can be drawn between classical fluctuation phenomena and fluctuations generated by the quantum dynamics. This helps us develop an intuition for the effects of quantum fluctuations. Also, mathematical techniques that were developed for analyzing Fokker-Plank equations in their traditional setting can be sequestered to help solve a quantum-mechanical problem.

There are, in fact, many ways in which to set up a quantum–classical correspondence. We will meet a number of these in this book and still more in Volume 2. The original ideas go back to the work of Wigner [3.2]. Wigner, however, was interested in general questions of quantum statistical mechanics, not specifically in quantum-optical applications; wide use of the methods of quantum–classical correspondence for problems in quantum optics only began with the work of Glauber [3.3] and Sudarshan [3.4]. These authors independently developed what is now commonly known as the Glauber–Sudarshan P representation, or simply the P representation, for the electromagnetic field. The representation is based on a correspondence in which *normal-ordered* operator averages are calculated as classical phase-space averages; it has been tailored for the special role played by normal-ordered averages in the theory of photodetection and quantum coherence [3.3, 3.5, 3.6]. The Wigner representation gives the averages of operators written in Weyl, or symmetric, order; other representations exist which use still different ordering conventions.

3.1 The Glauber–Sudarshan P Representation

The Glauber–Sudarshan P representation was introduced primarily for the description of statistical mixtures of coherent states – the closest approach within the quantum theory to the states of the electromagnetic field described by the classical statistical theory of optics. An understanding of this representation can therefore be built on a few simple properties of the coherent states. Formal definition of the P representation can, alternatively, be given without any mention of the coherent states; this is the more useful approach when we want to generalize the methods of quantum–classical correspondence to other representations for the field, and to representations for collections of two-level atoms. We will follow both routes in turn, to define the P representation and then illustrate its use by deriving a Fokker–Planck equation for the damped harmonic oscillator. We first follow the route based on coherent states, where we begin with a review of some of the more important properties of these states. Further discussion of the coherent states can be found in Louisell [3.7] and Sargent, Scully and Lamb [3.8].

3.1.1 Coherent States

The coherent state $|\alpha\rangle$ is the right eigenstate of the annihilation operator a with complex eigenvalue α:

$$a|\alpha\rangle = \alpha|\alpha\rangle, \qquad \langle\alpha|a^\dagger = (a|\alpha\rangle)^\dagger = \alpha^*\langle\alpha|. \tag{3.2}$$

From this definition we may prove the following properties of the coherent states:

Proposition 3.1 *If a harmonic oscillator, with Hamiltonian $H = \hbar\omega a^\dagger a$, has as its initial state the coherent state $|\alpha_0\rangle$, then it remains in a coherent state for all times with the oscillating complex amplitude $\alpha(t) = \alpha_0 e^{-i\omega t}$ – i.e. the time-dependent state of the oscillator is given by*

$$|\Psi(t)\rangle = e^{-(i/\hbar)Ht}|\alpha_0\rangle = e^{-i\omega_0 a^\dagger a t}|\alpha_0\rangle = |e^{-i\omega t}\alpha_0\rangle = |\alpha(t)\rangle. \tag{3.3}$$

Proof. We show that $|\Psi(t)\rangle$ is the right eigenstate of a with eigenvalue $\alpha(t)$:

$$\begin{aligned} a|\Psi(t)\rangle &= ae^{-i\omega a^\dagger a t}|\alpha_0\rangle \\ &= e^{-i\omega a^\dagger a t}\left(e^{i\omega a^\dagger a t} a e^{-i\omega a^\dagger a t}\right)|\alpha_0\rangle \\ &= \left(e^{-i\omega t}\alpha_0\right)\left(e^{-i\omega a^\dagger a t}|\alpha_0\rangle\right) \\ &= \alpha(t)|\Psi(t)\rangle, \end{aligned}$$

where we have used (1.40a) and (3.2). □

Proposition 3.2 *The coherent states are minimum uncertainty states: for a mechanical oscillator with position and momentum operators $\hat{q}$ and $\hat{p}$, respectively,*

$$\Delta q \Delta p = \sqrt{\left\langle(\hat{q} - \langle\hat{q}\rangle)^2\right\rangle}\sqrt{\left\langle(\hat{p} - \langle\hat{p}\rangle)^2\right\rangle} = \tfrac{1}{2}\hbar, \tag{3.4}$$

where the averages are taken with respect to a coherent state.

Proof. From (1.12a) and (1.12b),

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \tag{3.5a}$$

$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger). \tag{3.5b}$$

Then, for an oscillator in the state $|\alpha\rangle$,

$$\begin{aligned} \left\langle(\hat{q} - \langle\hat{q}\rangle)^2\right\rangle &= \langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2 \\ &= \frac{\hbar}{2m\omega}\langle\alpha|(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2})|\alpha\rangle - \langle\hat{q}\rangle^2 \\ &= \frac{\hbar}{2m\omega}\left[\langle\alpha|(aa^\dagger - a^\dagger a)|\alpha\rangle + (\alpha + \alpha^*)^2\right] - \langle\hat{q}\rangle^2 \\ &= \frac{\hbar}{2m\omega}\langle\alpha|[a, a^\dagger]|\alpha\rangle \\ &= \frac{\hbar}{2m\omega}, \end{aligned} \tag{3.6a}$$

where we have used (3.2) and the commutation relation (1.10); we assume that the state $|\alpha\rangle$ is normalized. Similarly,

$$\left\langle(\hat{p} - \langle\hat{p}\rangle)^2\right\rangle = \frac{\hbar m\omega}{2}. \tag{3.6b}$$

Thus,

$$\sqrt{\left\langle(\hat{q} - \langle\hat{q}\rangle)^2\right\rangle}\sqrt{\left\langle(\hat{p} - \langle\hat{p}\rangle)^2\right\rangle} = \tfrac{1}{2}\hbar.$$

□

Proposition 3.3 *A normalized coherent state can be expanded in terms of the Fock states $|n\rangle$, $n = 0, 1, 2, \ldots$, as*

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{3.7}$$

Proof. We write

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$$

and substitute this expansion into (3.2). Using $a|n\rangle = \sqrt{n}|n-1\rangle$, this gives the relationship

$$\sum_{n=1}^{\infty} c_n\sqrt{n}|n-1\rangle = \alpha\sum_{n=0}^{\infty} c_n|n\rangle.$$

Multiplying on the left by $\langle m|$ and using the orthogonality of the Fock states, we have

$$\sum_{n=1}^{\infty} c_n\sqrt{n}\,\delta_{m,n-1} = \alpha\sum_{n=0}^{\infty} c_n\delta_{m,n},$$

or

$$c_{m+1}\sqrt{m+1} = \alpha c_m;$$

thus,

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0.$$

c_0 is determined by the normalization condition $\langle\alpha|\alpha\rangle = 1$:

$$\begin{aligned}\langle\alpha|\alpha\rangle &= |c_0|^2 \sum_{n,m=0}^{\infty} \frac{\alpha^{*n}\alpha^m}{\sqrt{n!m!}}\langle n|m\rangle \\ &= |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \\ &= |c_0|^2 e^{|\alpha|^2};\end{aligned}$$

thus,

$$c_0 = e^{-\frac{1}{2}|\alpha|^2},$$

where the arbitrary phase has been chosen so that c_0 is real. □

Proposition 3.4 *The coherent states are not orthogonal; the overlap of the states $|\alpha\rangle$ and $|\beta\rangle$ is given by*

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}. \tag{3.8}$$

Note that $|\alpha\rangle$ and $|\beta\rangle$ are approximately orthogonal when $|\alpha-\beta|^2$ becomes large.

Proof. Using (3.7)

$$\begin{aligned}\langle\alpha|\beta\rangle &= e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} \sum_{n,m=0}^{\infty} \frac{\alpha^{*n}\beta^m}{\sqrt{n!m!}}\langle n|m\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{(\alpha^*\beta)^n}{n!} \\ &= e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} e^{\alpha^*\beta}.\end{aligned}$$

Then

$$\begin{aligned}|\langle\alpha|\beta\rangle|^2 &= e^{-|\alpha|^2} e^{-|\beta|^2} e^{\alpha^*\beta} e^{\alpha\beta^*} \\ &= e^{-|\alpha-\beta|^2}.\end{aligned}$$

□

Proposition 3.5 *The coherent states are complete:*

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = 1, \tag{3.9}$$

the integration being taken over the entire complex plane.

Proof. From (3.7),

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \frac{1}{\pi}\int d^2\alpha\, e^{-|\alpha|^2} \sum_{n,m=0}^{\infty} \frac{\alpha^{*n}\alpha^m}{\sqrt{n!m!}} |n\rangle\langle m|,$$

or, in polar coordinates,

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \frac{1}{\pi}\sum_{n,m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{n!m!}} \int_0^{\infty} dr\, e^{-r^2} r^{n+m+1} \int_0^{2\pi} d\phi\, e^{-i(n-m)\phi},$$

where $\alpha = re^{i\phi}$. The integration over ϕ gives zero unless n is equal to m. Thus,

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = 2\sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \int_0^{\infty} dr\, e^{-r^2} r^{2n+1}.$$

After integrating by parts n times,

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = 2\sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!}\frac{1}{2}n! = \sum_{n=0}^{\infty} |n\rangle\langle n| = 1.$$

The final step follows from the completeness of the Fock states. □

Proposition 3.6 *The coherent states can be generated from the vacuum state by the action of the creation operator $a^\dagger$:*

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger}|0\rangle. \tag{3.10}$$

Proof. Using $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, we have

$$\begin{aligned}e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger}|0\rangle &= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} a^{\dagger n}|0\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\sqrt{n!}\,|n\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle.\end{aligned}$$

This is the expression (3.7) for the Fock state expansion of the coherent state $|\alpha\rangle$. □

3.1.2 Diagonal Representation for the Density Operator Using Coherent States

Using the completeness of the Fock states, a representation for the density operator ρ in terms of these states is obtained by multiplying on the left and right by the unit operator expressed as a sum of outer products:

$$\begin{aligned}\rho &= \left(\sum_{n=0}^{\infty}|n\rangle\langle n|\right)\rho\left(\sum_{m=0}^{\infty}|m\rangle\langle m|\right)\\ &= \sum_{n,m=0}^{\infty}\rho_{n,m}|n\rangle\langle m|,\end{aligned} \tag{3.11}$$

with $\rho_{n,m} \equiv \langle n|\rho|m\rangle$. The Fock states are orthogonal as well as being complete, as is the common situation for a set of basis states. The coherent states are not orthogonal (Proposition 3.4). However, they are complete (Proposition 3.5), and this is all we need to define a representation for ρ analogous to (3.11). From (3.9), we may write

$$\begin{aligned}\rho &= \left(\frac{1}{\pi}\int d^2\alpha\,|\alpha\rangle\langle\alpha|\right)\rho\left(\frac{1}{\pi}\int d^2\beta\,|\beta\rangle\langle\beta|\right)\\ &= \frac{1}{\pi^2}\int d^2\alpha\int d^2\beta\,|\alpha\rangle\langle\beta|\langle\alpha|\rho|\beta\rangle.\end{aligned} \tag{3.12}$$

Glauber has defined what he calls the R *representation*, expanding the density operator in the form [3.3]

$$\rho = \frac{1}{\pi^2}\int d^2\alpha\int d^2\beta\,|\alpha\rangle\langle\beta|\,e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{1}{2}|\beta|^2}R(\alpha^*,\beta), \tag{3.13}$$

where

$$\begin{aligned}R(\alpha^*,\beta) &\equiv e^{\frac{1}{2}|\alpha|^2}e^{\frac{1}{2}|\beta|^2}\langle\alpha|\rho|\beta\rangle\\ &= e^{\frac{1}{2}|\alpha|^2}e^{\frac{1}{2}|\beta|^2}\left(e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\langle n|\right)\rho\left(e^{-\frac{1}{2}|\beta|^2}\sum_{m=0}^{\infty}\frac{\beta^m}{\sqrt{m!}}|m\rangle\right)\\ &= \sum_{n,m=0}^{\infty}\frac{\alpha^{*n}\beta^m}{\sqrt{n!m!}}\rho_{n,m}.\end{aligned} \tag{3.14}$$

Clearly, this representation follows the familiar methods for specifying an operator in terms of its matrix elements; the exponential factors appearing in (3.13) merely simplify the relationship between the function $R(\alpha^*,\beta)$ and the Fock state matrix elements $\rho_{n,m}$. The P representation is rather different.

The *Glauber–Sudarshan P representation* relies on the fact that the coherent states are not orthogonal. In technical terms they then form an overcomplete basis, and, as a consequence, it is possible to expand ρ as a *diagonal* sum over coherent states:

$$\rho = \int d^2\alpha\,|\alpha\rangle\langle\alpha|P(\alpha). \tag{3.15}$$

This representation for ρ is appealing because the function $P(\alpha)$ plays a role rather analogous to that of a classical probability distribution. First, note that

$$\begin{aligned}\int d^2\alpha\,P(\alpha) &= \int d^2\alpha\,\langle\alpha|\alpha\rangle P(\alpha)\\ &= \mathrm{tr}\left(\int d^2\alpha\,|\alpha\rangle\langle\alpha|P(\alpha)\right)\\ &= \mathrm{tr}(\rho)\\ &= 1,\end{aligned} \tag{3.16}$$

where we have inserted $\langle\alpha|\alpha\rangle = 1$ and used the cyclic property of the trace. Thus, $P(\alpha)$ is normalized like a classical probability distribution. Note also that for the expectation values of operators written in normal order (creation operators to the left and annihilation operators to the right), on substituting the expansion (3.15) for ρ,

$$\begin{aligned}\langle a^{\dagger p}a^q\rangle &\equiv \mathrm{tr}\big(\rho a^{\dagger p}a^q\big)\\ &= \mathrm{tr}\left(\int d^2\alpha\,|\alpha\rangle\langle\alpha|P(\alpha)a^{\dagger p}a^q\right)\\ &= \int d^2\alpha\,P(\alpha)\langle\alpha|a^{\dagger p}a^q|\alpha\rangle\\ &= \int d^2\alpha\,P(\alpha)\alpha^{*p}\alpha^q.\end{aligned} \tag{3.17}$$

Normal-ordered averages are therefore calculated in the way that averages are calculated in classical statistics, with $P(\alpha)$ playing the role of the probability distribution [(3.16) is a special case of this result with $p = q = 0$]. We will introduce the notation

$$\big(\overline{\alpha^{*p}\alpha^q}\big)_P \equiv \int d^2\alpha\,P(\alpha)\alpha^{*p}\alpha^q, \tag{3.18}$$

and write

$$\langle a^{\dagger p}a^q\rangle = \big(\overline{\alpha^{*p}\alpha^q}\big)_P. \tag{3.19}$$

As mentioned earlier, obtaining normal-ordered averages in this way is particularly useful because measurements in quantum optics have a direct relationship to such normal-ordered quantities, a consequence of the fact that photoelectric detectors work by the absorption of photons.

The analogy between $P(\alpha)$ and a classical probability distribution over coherent states must be made with reservation, however. In the Fock-state representation $\rho_{n,n} = \langle n|\rho|n\rangle$ is an actual probability; it is the probability that the oscillator will be found in the state $|n\rangle$ – the probability that the field mode will be found to contain n photons. But because of the orthogonality of the Fock states, only a limited class of states can be represented by the diagonal matrix elements $\rho_{n,n}$ alone. There exist states whose complete representation requires that at least some nonzero numbers $\rho_{n,m} = \langle n|\rho|m\rangle$, $n \neq m$, be specified in addition to the probabilities $\rho_{n,n}$. The coherent states are not orthogonal, and it is therefore possible to make a diagonal expansion for ρ that is not restricted in the same way; the expansion (3.15) does not automatically require that the off-diagonal coherent state matrix elements vanish. With the help of (3.8), from (3.15) we obtain

$$\begin{aligned}\langle\alpha|\rho|\beta\rangle &= \int d^2\lambda\, \langle\alpha|\lambda\rangle\langle\lambda|\beta\rangle P(\lambda)\\ &= \int d^2\lambda\, e^{-\frac{1}{2}|\lambda-\alpha|^2} e^{-\frac{1}{2}|\lambda-\beta|^2} P(\lambda). \qquad (3.20)\end{aligned}$$

There is no need for this to vanish when $\alpha \neq \beta$. There is a price to pay for this versatility, however. We must now accept that $P(\alpha)$ is not strictly a probability. When $\alpha = \beta$, (3.20) gives

$$\langle\alpha|\rho|\alpha\rangle = \int d^2\lambda\, e^{-|\lambda-\alpha|^2} P(\lambda). \qquad (3.21)$$

Since $e^{-|\lambda-\alpha|^2}$ is not a δ-function, $\langle\alpha|\rho|\alpha\rangle \neq P(\alpha)$. Only when $P(\lambda)$ is sufficiently broad compared to the Gaussian filter inside the integral in (3.21) does it approximate a probability. Also, although the probability $\langle\alpha|\rho|\alpha\rangle$ must be positive, (3.21) does not require $P(\alpha)$ to be so. Thus, unlike a classical probability, $P(\alpha)$ can take negative values over a limited range [although (3.16) must still be satisfied]. $P(\alpha)$ is not, therefore, a probability distribution, and for this reason it is often referred to as a *quasi*distribution function. We will simply use the word "distribution". In fact, this is quite correct usage if "distribution" is interpreted in the sense of generalized functions. We will see shortly that $P(\alpha)$ is, most generally, a generalized function.

3.1.3 Examples: Coherent States, Thermal States, and Fock States

It is clear from (3.15) that the coherent state $|\alpha_0\rangle$ – density operator $\rho = |\alpha_0\rangle\langle\alpha_0|$ – is represented by the P distribution

$$P(\alpha) = \delta^{(2)}(\alpha-\alpha_0) \equiv \delta(x-x_0)\delta(y-y_0), \qquad (3.22)$$

where $\alpha = x+iy$ and $\alpha_0 = x_0+iy_0$. Can we find a diagonal representation for any density operator? To answer this question we must try to invert (3.15). This is made possible using the relationship

$$\begin{aligned}\mathrm{tr}\Big(\rho e^{iz^*a^\dagger}e^{iza}\Big) &= \mathrm{tr}\left\{\left[\int d^2\alpha\, |\alpha\rangle\langle\alpha|P(\alpha)\right] e^{iz^*a^\dagger}e^{iza}\right\}\\ &= \int d^2\alpha\, P(\alpha)\langle\alpha|e^{iz^*a^\dagger}e^{iza}|\alpha\rangle\\ &= \int d^2\alpha\, P(\alpha)e^{iz^*\alpha^*}e^{iz\alpha}. \qquad (3.23)\end{aligned}$$

Equation (3.23) is just a two-dimensional Fourier transform. The inverse transform gives

$$P(\alpha) = \frac{1}{\pi^2}\int d^2z\, \mathrm{tr}\Big(\rho e^{iz^*a^\dagger}e^{iza}\Big) e^{-iz^*\alpha^*}e^{-iz\alpha}. \qquad (3.24)$$

Thus, if the Fourier transform of the function defined by the trace in (3.24) exists for a given density operator ρ, we have our P distribution representing that density operator. A general expression for $P(\alpha)$ in terms of the Fock-state representation of ρ follows by substituting (3.11) into (3.24) and using the cyclic property of the trace:

$$\begin{aligned}P(\alpha) &= \frac{1}{\pi^2}\int d^2z\left(\sum_{n,m=0}^{\infty}\rho_{n,m}\langle m|e^{iz^*a^\dagger}e^{iza}|n\rangle\right)e^{-iz^*\alpha^*}e^{-iz\alpha}\\ &= \frac{1}{\pi^2}\int d^2z\left(\sum_{n,m=0}^{\infty}\sum_{n',m'=0}^{\infty}\rho_{n,m}\langle m|\frac{(iz^*a^\dagger)^{m'}}{m'!}\frac{(iza)^{n'}}{n'!}|n\rangle\right)\\ &\quad\times e^{-iz^*\alpha^*}e^{-iz\alpha}\\ &= \frac{1}{\pi^2}\int d^2z\left(\sum_{n=0}^{\infty}\sum_{n'=0}^{n}\sum_{m=0}^{\infty}\sum_{m'=0}^{m}\rho_{n,m}\frac{(iz^*)^{m'}}{m'!}\sqrt{\frac{m!}{(m-m')!}}\right.\\ &\quad\left.\times\frac{(iz)^{n'}}{n'!}\sqrt{\frac{n!}{(n-n')!}}\,\delta_{n-n',m-m'}\right)e^{-iz^*\alpha^*}e^{-iz\alpha}.\end{aligned}$$

Noting that

$$\sum_{n=0}^{\infty}\sum_{n'=0}^{n}\sum_{m=0}^{\infty}\sum_{m'=0}^{m}\cdots \equiv \sum_{n'=0}^{\infty}\sum_{n-n'=0}^{\infty}\sum_{m'=0}^{\infty}\sum_{m-m'=0}^{\infty}\cdots,$$

and changing the summation indices, with $n' \to n$, $m' \to m$, and $n-n' = m-m' \to k$, we find

$$\begin{aligned}P(\alpha) &= \frac{1}{\pi^2}\int d^2z\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\rho_{n+k,m+k}\frac{\sqrt{(n+k)!}\sqrt{(m+k)!}}{k!}\right.\\ &\quad\left.\times\frac{(iz^*)^m}{m!}\frac{(iz)^n}{n!}\right)e^{-iz^*\alpha^*}e^{-iz\alpha}. \qquad (3.25)\end{aligned}$$

Exercise 3.1 Substitute $\rho = |\alpha_0\rangle\langle\alpha_0|$ into (3.24) and the Fock-state representation for this density operator into (3.25); show that both of these equations reproduce the P distribution (3.22) for the coherent state. For the thermal state

$$\rho = (1 - e^{-\hbar\omega/k_BT})e^{-\hbar\omega a^\dagger a/k_BT}, \tag{3.26}$$

show that (3.25) gives

$$\begin{aligned} P(\alpha) &= \frac{1}{\pi^2}\int d^2z\, e^{-|z|^2\langle\hat{n}\rangle} e^{-iz^*\alpha^*} e^{-iz\alpha} \\ &= \frac{1}{\pi\langle\hat{n}\rangle}\exp\left(-\frac{|\alpha|^2}{\langle\hat{n}\rangle}\right), \end{aligned} \tag{3.27}$$

where

$$\langle\hat{n}\rangle \equiv \langle a^\dagger a\rangle = \frac{e^{-\hbar\omega/k_BT}}{1 - e^{-\hbar\omega/k_BT}}. \tag{3.28}$$

Now, consider the P distribution representing a Fock state. We will take $\rho = |l\rangle\langle l|$ where l can be any non-negative integer. From (3.25),

$$\begin{aligned} P(\alpha) &= \frac{1}{\pi^2}\int d^2z\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\delta_{n+k,l}\delta_{m+k,l}\frac{l!}{k!}\frac{(iz^*)^m}{m!}\frac{(iz)^n}{n!}\right) \\ &\quad\times e^{-iz^*\alpha^*}e^{-iz\alpha} \\ &= \frac{1}{\pi^2}\int d^2z\left(\sum_{k=0}^{l}\frac{(-1)^k|z|^{2k}}{k!}\frac{l!}{k!(l-k)!}\right)e^{-iz^*\alpha^*}e^{-iz\alpha}, \end{aligned} \tag{3.29}$$

where we have changed the summation index, with $l - k \to k$. Since the summation in (3.29) does not extend to infinity, the expression inside the bracket is a polynomial, and it clearly diverges for $|z| \to \infty$. Thus, this Fourier transform does not exist in the ordinary sense; it would appear that we cannot represent a Fock state using only a diagonal expansion in coherent states. If, however, we write

$$\delta^{(2)}(\alpha) \equiv \frac{1}{\pi^2}\int d^2z\, e^{-iz^*\alpha^*}e^{-iz\alpha} \tag{3.30}$$

and use the ordinary rules of differentiation inside the integral in (3.29), we may evaluate the integral in terms of derivatives of the δ-function. This gives the P distribution

$$P(\alpha) = \sum_{k=0}^{l}\frac{l!}{k!(l-k)!}\frac{1}{k!}\frac{\partial^{2k}}{\partial\alpha^k\partial\alpha^{*k}}\delta^{(2)}(\alpha). \tag{3.31}$$

Note 3.1 We will have many occasions to take derivatives with respect to complex conjugate variables. It is convenient to do this by reading the complex variable and its conjugate as two independent variables. This is allowed because

$$\frac{\partial}{\partial\alpha}\alpha^* = \left(\frac{\partial}{\partial\alpha^*}\alpha\right)^* = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x - iy) = \frac{1}{2}\left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}y\right) = 0, \tag{3.32a}$$

and, of course,

$$\frac{\partial}{\partial\alpha}\alpha = \left(\frac{\partial}{\partial\alpha^*}\alpha^*\right)^* = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x + iy) = \frac{1}{2}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right) = 1. \tag{3.32b}$$

The mathematical theory that gives precise meaning to (3.31) is the theory of *generalized functions* [3.9–3.11] or *distributions* (in the technical sense of "Schwartz distributions" and "tempered distributions" [3.12, 3.13]). Within this theory the Fourier transform can be formally generalized to cover nonintegrable functions such as polynomials. Such Fourier transforms are not functions in the usual sense; (3.31) does not tell us how to associate a number, $P(\alpha)$, with each value of the variable α. There is certainly no way, then, to interpret $P(\alpha)$ as a probability distribution. It is, however, a "distribution" in the sense defined by the theory of generalized functions. There is no need for us to get deeply involved with the formal theory of generalized functions. Those interested can study this in the books by Lighthill [3.11] and Bremermann [3.13]. Nevertheless, in order to appreciate the sense in which (3.31) provides a diagonal representation for the Fock states we should spend just a little time refreshing our memories about some of the basic properties of generalized functions.

Generalized functions "live" inside integrals. There, they are integrated against some ordinary function from a space of *test functions*. The value of the integral for a given test function is defined as the limit of a sequence of integrals obtained by replacing the generalized function by a sequence of ordinary well-behaved functions. The generalized function is then, in this sense, the limit of a sequence of ordinary functions. Of course, the sequence of functions defining a given generalized function is not unique. For example, for a suitable class of test functions, the δ-function acts inside an integral as the limit of a sequence of Gaussians:

$$\delta(x) \equiv \lim_{n\to\infty}\sqrt{\frac{n}{\pi}}e^{-nx^2}, \tag{3.33}$$

where the strict sense of this statement is

$$\int_{-\infty}^{\infty} dx\, \delta(x)\phi(x) \equiv \lim_{n\to\infty} \int_{-\infty}^{\infty} dx\, \sqrt{\frac{n}{\pi}} e^{-nx^2} \phi(x) = \phi(0). \tag{3.34}$$

Here, the test function $\phi(x)$ must be continuous and grow more slowly at infinity than $Ce^{a|x|}$, with C and a constants. A sequence of functions that decrease faster than Gaussians at infinity would allow us to define the δ-function on a larger space of test functions; most generally, for all continuous functions. Thus, in formal language, generalized functions operate as *functionals*; they associate a number (the limiting value of a sequence of ordinary integrals) with each function from a space of test functions.

The derivative of a generalized function is also a generalized function, defined via the rules of partial integration. Taking $\phi(x) = \psi'(x)$ in (3.34), we can write

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx\, \delta(x)\psi'(x) \\
&\quad = \lim_{n\to\infty} \int_{-\infty}^{\infty} dx\, \sqrt{\frac{n}{\pi}} e^{-nx^2} \psi'(x) \\
&\quad = \lim_{n\to\infty} \left[\sqrt{\frac{n}{\pi}} e^{-nx^2} \psi(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx \left(-2nx\sqrt{\frac{n}{\pi}} e^{-nx^2} \right) \psi(x) \right] \\
&\quad = -\lim_{n\to\infty} \int_{-\infty}^{\infty} dx \left(-2nx\sqrt{\frac{n}{\pi}} e^{-nx^2} \right) \psi(x).
\end{aligned} \tag{3.35}$$

Then, if $\delta'(x)$ is the generalized function defined by the sequence of functions obtained as the derivative of the sequence defining $\delta(x)$ – the functions inside the bracket in (3.35) – the formula for partial integration is preserved:

$$\int_{-\infty}^{\infty} dx\, \delta'(x)\psi(x) = -\int_{-\infty}^{\infty} dx\, \delta(x)\psi'(x) = -\psi'(0). \tag{3.36}$$

More generally, for the nth derivative of the δ-function, $\delta^{(n)}(x)$, we have

$$\int_{-\infty}^{\infty} dx\, \delta^{(n)}(x)\psi(x) = (-1)^n \int_{-\infty}^{\infty} dx\, \delta(x)\psi^{(n)}(x) = (-1)^n \psi^{(n)}(0), \tag{3.37}$$

where $\psi^{(n)}(x)$ is the nth derivative of $\psi(x)$. [Do not confuse the notation for the nth derivative of the δ-function with the notation $\delta^{(2)}(\alpha)$ for the two-dimensional δ-function.]

Let us now use (3.37) to see explicitly how (3.31) provides a diagonal representation for the Fock states. We will consider the one-photon state, the simplest example; the general case can be done as an exercise. For $l = 1$, from (3.31),

$$P(\alpha) = \delta^{(2)}(\alpha) + \frac{\partial^2}{\partial\alpha\partial\alpha^*}\delta^{(2)}(\alpha).$$

Substituting into the diagonal expansion (3.15), and using (3.37) (twice for the two-dimensional δ-function),

$$\begin{aligned}
\rho &= \int d^2\alpha\, |\alpha\rangle\langle\alpha| P(\alpha) \\
&= \int d^2\alpha\, |\alpha\rangle\langle\alpha| \left[\delta^{(2)}(\alpha) + \frac{\partial^2}{\partial\alpha\partial\alpha^*}\delta^{(2)}(\alpha) \right] \\
&= |0\rangle\langle 0| + \int d^2\alpha \left(\frac{\partial^2}{\partial\alpha\partial\alpha^*} |\alpha\rangle\langle\alpha| \right) \delta^{(2)}(\alpha) \\
&= |0\rangle\langle 0| + \frac{\partial^2}{\partial\alpha\partial\alpha^*} |\alpha\rangle\langle\alpha| \Big|_{\alpha=0}.
\end{aligned} \tag{3.38}$$

From this we must recover $\rho = |1\rangle\langle 1|$. Using (3.10), we note that

$$\begin{aligned}
\frac{\partial}{\partial\alpha}|\alpha\rangle\langle\alpha| &= \frac{\partial}{\partial\alpha}\left(e^{-|\alpha|^2} e^{\alpha a^\dagger} |0\rangle\langle 0| e^{\alpha^* a} \right) \\
&= (a^\dagger - \alpha^*)|\alpha\rangle\langle\alpha|,
\end{aligned} \tag{3.39a}$$

$$\begin{aligned}
\frac{\partial}{\partial\alpha^*}|\alpha\rangle\langle\alpha| &= \frac{\partial}{\partial\alpha^*}\left(e^{-|\alpha|^2} e^{\alpha a^\dagger} |0\rangle\langle 0| e^{\alpha^* a} \right) \\
&= |\alpha\rangle\langle\alpha|(a - \alpha).
\end{aligned} \tag{3.39b}$$

Then (3.38) readily gives the required result:

$$\begin{aligned}
\rho &= |0\rangle\langle 0| + \frac{\partial}{\partial\alpha}\left[|\alpha\rangle\langle\alpha|(a - \alpha) \right] \Big|_{\alpha=0} \\
&= |0\rangle\langle 0| + \left[(a^\dagger - \alpha^*)|\alpha\rangle\langle\alpha|(a - \alpha) - |\alpha\rangle\langle\alpha| \right] \Big|_{\alpha=0} \\
&= |0\rangle\langle 0| + \left(a^\dagger|0\rangle\langle 0|a - |0\rangle\langle 0| \right) \\
&= |1\rangle\langle 1|.
\end{aligned}$$

Exercise 3.2 Equation (3.31) is not always the most convenient form to use in calculations. Show that $P(\alpha)$ for the Fock state $|l\rangle$ takes the alternate forms

$$P(\alpha) = \frac{1}{l!} e^{|\alpha|^2} \frac{\partial^{2l}}{\partial\alpha^l \partial\alpha^{*l}} \delta^{(2)}(\alpha), \tag{3.40}$$

and in polar coordinates, with $\alpha = re^{i\theta}$,

$$P(\alpha) = \frac{1}{2\pi r} \frac{l!}{(2l)!} e^{r^2} \frac{\partial^{2l}}{\partial r^{2l}} \delta(r). \tag{3.41}$$

Show that both of these expressions give $\rho = |l\rangle\langle l|$ when substituted into the diagonal expansion for ρ [Eq. (3.15)].

Applications of the P representation in quantum optics have largely been restricted to situations in which $P(\alpha)$ exists as an ordinary function, as it does, for example, for a thermal state [Eq. (3.27)]. With the use of generalized functions it is actually possible to give any density operator a diagonal representation [3.14, 3.15]. As we stated earlier, however, our main objective when introducing the quantum–classical correspondence is to cast the quantum-mechanical theory into a form closely analogous to a classical statistical theory. $P(\alpha)$ is never strictly a probability for observing the coherent state $|\alpha\rangle$, but it can take the *form* of a probability distribution, and when it does, this can be used to aid our intuition – as an example, the phase-independent distribution given by (3.27) essentially corresponds to the classical picture of a field mode subject to thermal fluctuations. Our intuition finds little assistance from a representation in terms of a generalized function. The value of preserving the analogy with a classical statistical system will be further underlined as we now use the P representation to describe the dynamics of the damped harmonic oscillator.

3.1.4 Fokker–Planck Equation for the Damped Harmonic Oscillator

In Sect. 1.4.1 we derived the master equation for the damped harmonic oscillator:

$$\begin{aligned}\dot{\rho} = -i\omega_0[a^\dagger a, \rho] &+ \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)\\ &+ \gamma\bar{n}(a\rho a^\dagger + a^\dagger\rho a - a^\dagger a\rho - \rho a a^\dagger).\end{aligned} \tag{3.42}$$

Our goal in this section is to substitute the diagonal representation (3.15) for ρ, and convert the operator master equation into an equation of motion for P. Obviously, we must assume the existence of a time-dependent P distribution, $P(\alpha, t)$, to represent ρ at each instant t.

After substituting for ρ, (3.42) becomes

$$\begin{aligned}&\int d^2\alpha\, |\alpha\rangle\langle\alpha| \frac{\partial}{\partial t} P(\alpha, t)\\ &\quad= \int d^2\alpha\, P(\alpha, t)\big[-i\omega_0\big(a^\dagger a|\alpha\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|a^\dagger a\big)\\ &\qquad+ \frac{\gamma}{2}\big(2a|\alpha\rangle\langle\alpha|a^\dagger - a^\dagger a|\alpha\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|a^\dagger a\big)\\ &\qquad+ \gamma\bar{n}\big(a|\alpha\rangle\langle\alpha|a^\dagger + a^\dagger|\alpha\rangle\langle\alpha|a - a^\dagger a|\alpha\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|aa^\dagger\big)\big].\end{aligned} \tag{3.43}$$

The central step in our derivation is to replace the action of the operators a and $a^\dagger$ on $|\alpha\rangle\langle\alpha|$ (both to the right and to the left) by multiplication by the complex variables α and α^*, and the action of partial derivatives with respect to these variables. This can be accomplished using (3.2) and (3.39):

$$a|\alpha\rangle\langle\alpha|a^\dagger = \alpha|\alpha\rangle\langle\alpha|\alpha^* = |\alpha|^2|\alpha\rangle\langle\alpha|, \tag{3.44a}$$

$$a^\dagger a|\alpha\rangle\langle\alpha| = a^\dagger\alpha|\alpha\rangle\langle\alpha| = \alpha a^\dagger|\alpha\rangle\langle\alpha| = \alpha\left(\frac{\partial}{\partial\alpha} + \alpha^*\right)|\alpha\rangle\langle\alpha|, \tag{3.44b}$$

$$|\alpha\rangle\langle\alpha|a^\dagger a = |\alpha\rangle\langle\alpha|\alpha^* a = \alpha^*|\alpha\rangle\langle\alpha|a = \alpha^*\left(\frac{\partial}{\partial\alpha^*} + \alpha\right)|\alpha\rangle\langle\alpha|, \tag{3.44c}$$

$$|\alpha\rangle\langle\alpha|aa^\dagger = \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)|\alpha\rangle\langle\alpha|a^\dagger = \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)\alpha^*|\alpha\rangle\langle\alpha|, \tag{3.44d}$$

$$a^\dagger|\alpha\rangle\langle\alpha|a = \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)|\alpha\rangle\langle\alpha|a = \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)\left(\frac{\partial}{\partial\alpha^*} + \alpha\right)|\alpha\rangle\langle\alpha|. \tag{3.44e}$$

Using these results in (3.43), after some cancelation, we find

$$\begin{aligned}\int d^2\alpha\, |\alpha\rangle\langle\alpha|\frac{\partial}{\partial t}P(\alpha, t) = \int d^2\alpha\, P(\alpha, t)\Big[&-\Big(\frac{\gamma}{2} + i\omega_0\Big)\alpha\frac{\partial}{\partial\alpha}\\ &-\Big(\frac{\gamma}{2} - i\omega_0\Big)\alpha^*\frac{\partial}{\partial\alpha^*} + \gamma\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\Big]|\alpha\rangle\langle\alpha|.\end{aligned} \tag{3.45}$$

It is a short step to an equation of motion for P. The partial derivatives which now act to the right on $|\alpha\rangle\langle\alpha|$ can be transferred to the distribution $P(\alpha, t)$ by integrating by parts. We will assume that $P(\alpha, t)$ vanishes sufficiently rapidly at infinity to allow us to drop the boundary terms. Then (3.45) becomes

$$\begin{aligned}\int d^2\alpha\, |\alpha\rangle\langle\alpha|\frac{\partial}{\partial t}P(\alpha, t) = \int d^2\alpha\, |\alpha\rangle\langle\alpha|\Big[&\Big(\frac{\gamma}{2} + i\omega_0\Big)\frac{\partial}{\partial\alpha}\alpha\\ &+\Big(\frac{\gamma}{2} - i\omega_0\Big)\frac{\partial}{\partial\alpha^*}\alpha^* + \gamma\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\Big]P(\alpha, t).\end{aligned} \tag{3.46}$$

Note 3.2 When integrating by parts α and α^* may be read as independent variables, as in differentiation (Note 3.1). Explicitly, for given functions $f(\alpha)$ and $g(\alpha)$ (whose product vanishes at infinity),

$$\begin{aligned}&\int d^2\alpha\, f(\alpha)\frac{\partial}{\partial\alpha}g(\alpha)\\ &\quad= \int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} dy\, f(x, y)\frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)g(x, y)\\ &\quad= \frac{1}{2}\int_{-\infty}^{\infty} dy\Big[f(x, y)g(x, y)\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, g(x, y)\frac{\partial}{\partial x}f(x, y)\Big]\\ &\qquad- i\frac{1}{2}\int_{-\infty}^{\infty} dx\Big[f(x, y)g(x, y)\Big|_{y=-\infty}^{\infty} - \int_{-\infty}^{\infty} dy\, g(x, y)\frac{\partial}{\partial y}f(x, y)\Big]\end{aligned}$$

$$= -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, g(x,y) \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f(x,y)$$
$$= -\int d^2\alpha\, g(\alpha) \frac{\partial}{\partial \alpha} f(\alpha).$$

Similarly,

$$\int d^2\alpha\, f(\alpha) \frac{\partial}{\partial \alpha^*} g(\alpha) = -\int d^2\alpha\, g(\alpha) \frac{\partial}{\partial \alpha^*} f(\alpha).$$

A *sufficient condition* for (3.46) to be satisfied is that the P distribution obeys the equation of motion

$$\frac{\partial P}{\partial t} = \left[\left(\frac{\gamma}{2} + i\omega_0\right)\frac{\partial}{\partial \alpha}\alpha + \left(\frac{\gamma}{2} - i\omega_0\right)\frac{\partial}{\partial \alpha^*}\alpha^* + \gamma\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]P. \tag{3.47}$$

We have replaced the operator equation (3.42) by a partial differential equation for P. This is the *Fokker–Planck equation for the damped harmonic oscillator in the P representation.*

Exercise 3.3 The question arises as to whether (3.47) is a *necessary condition* for (3.46) to be satisfied. Multiply both sides of (3.46) on the left by $e^{iz^* a^\dagger} e^{iza}$ and take the trace to show that the necessary condition is that the Fourier transforms of both sides of (3.47) are equal.

3.1.5 Solution of the Fokker–Planck Equation

We will discuss the properties of Fokker–Planck equations in detail in Chap. 5. For the present let us simply illustrate how (3.47) describes the damped harmonic oscillator. We will solve this equation for an initial coherent state $|\alpha_0\rangle$. Thus, we seek the Green function $P(\alpha, \alpha^*, t|\alpha_0, \alpha_0^*, 0)$, with initial condition

$$P(\alpha, \alpha^*, 0|\alpha_0, \alpha_0^*, 0) = \delta^{(2)}(\alpha - \alpha_0) \equiv \delta(x - x_0)\delta(y - y_0). \tag{3.48}$$

From now on we display P with two complex conjugate arguments consistent with the interpretation of derivatives and integrals explained below (3.31) and (3.46).

It is convenient to transform to a frame rotating at the frequency ω_0, with

$$\alpha = e^{-i\omega_0 t}\tilde{\alpha}, \qquad \alpha^* = e^{i\omega_0 t}\tilde{\alpha}^*, \tag{3.49}$$

and

$$P(\alpha, \alpha^*, t) = \tilde{P}(\tilde{\alpha}, \tilde{\alpha}^*, t). \tag{3.50}$$

We have

$$\frac{\partial \tilde{P}}{\partial t} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial P}{\partial \alpha^*}\frac{\partial \alpha^*}{\partial t}$$
$$= \frac{\partial P}{\partial t} - i\omega_0\left(\alpha\frac{\partial P}{\partial \alpha} - \alpha^*\frac{\partial P}{\partial \alpha^*}\right)$$
$$= \frac{\partial P}{\partial t} - i\omega_0\left(\frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \alpha^*}\alpha^*\right)P. \tag{3.51}$$

After substituting for $\partial P/\partial t$ from (3.47),

$$\frac{\partial \tilde{P}}{\partial t} = \left[\frac{\gamma}{2}\left(\frac{\partial}{\partial\tilde{\alpha}}\tilde{\alpha} + \frac{\partial}{\partial\tilde{\alpha}^*}\tilde{\alpha}^*\right) + \gamma\bar{n}\frac{\partial^2}{\partial\tilde{\alpha}\partial\tilde{\alpha}^*}\right]\tilde{P}, \tag{3.52}$$

or, in terms of the real and imaginary parts of $\tilde{\alpha}$,

$$\frac{\partial \tilde{P}}{\partial t} = \left[\frac{\gamma}{2}\left(\frac{\partial}{\partial\tilde{x}}\tilde{x} + \frac{\partial}{\partial\tilde{y}}\tilde{y}\right) + \frac{\gamma\bar{n}}{4}\left(\frac{\partial^2}{\partial\tilde{x}^2} + \frac{\partial^2}{\partial\tilde{y}^2}\right)\right]\tilde{P}, \tag{3.53}$$

where $\tilde{\alpha} = \tilde{x} + i\tilde{y}$. Solutions can now be sought using separation of variables. We write

$$\tilde{P}(\tilde{x}, \tilde{y}, t) = X(\tilde{x}, t)Y(\tilde{y}, t), \tag{3.54}$$

where the functions X and Y satisfy the independent equations

$$\frac{\partial X}{\partial t} = \left(\frac{\gamma}{2}\frac{\partial}{\partial\tilde{x}}\tilde{x} + \frac{\gamma\bar{n}}{4}\frac{\partial^2}{\partial\tilde{x}^2}\right)X, \tag{3.55a}$$

$$\frac{\partial Y}{\partial t} = \left(\frac{\gamma}{2}\frac{\partial}{\partial\tilde{y}}\tilde{y} + \frac{\gamma\bar{n}}{4}\frac{\partial^2}{\partial\tilde{y}^2}\right)Y. \tag{3.55b}$$

These are to be solved for $X(\tilde{x}, t|\tilde{x}_0, 0)$ and $Y(\tilde{y}, t|\tilde{y}_0, 0)$, subject to the initial conditions

$$X(\tilde{x}, 0|\tilde{x}_0, 0) = \delta(\tilde{x} - \tilde{x}_0), \tag{3.56a}$$
$$Y(\tilde{y}, 0|\tilde{y}_0, 0) = \delta(\tilde{y} - \tilde{y}_0). \tag{3.56b}$$

Consider (3.55a). Its solution is found by taking the Fourier transform on both sides of the equation. We find

$$\frac{\partial U}{\partial t} = -\left(\frac{\gamma}{2}\tilde{u}\frac{\partial}{\partial\tilde{u}} + \frac{\gamma\bar{n}}{4}\tilde{u}^2\right)U, \tag{3.57}$$

where

$$U(\tilde{u}, t|\tilde{x}_0, 0) = \int_{-\infty}^{\infty} d\tilde{x}\, X(\tilde{x}, t|\tilde{x}_0, 0)e^{i\tilde{x}\tilde{u}}, \tag{3.58}$$

and, from (3.56a), the initial condition for U is

$$U(\tilde{u}, 0|\tilde{x}_0, 0) = e^{i\tilde{x}_0\tilde{u}}. \tag{3.59}$$

We then solve (3.57) by the method of characteristics [3.16]. The subsidiary equations are

$$\frac{dt}{1}=\frac{d\tilde{u}}{(\gamma/2)\tilde{u}}=\frac{dU}{-(\gamma\bar{n}/4)\tilde{u}^2U}\,, \tag{3.60}$$

with solutions

$$\tilde{u}e^{-(\gamma/2)t}=\text{constant}\,, \tag{3.61a}$$

$$Ue^{(\bar{n}/4)\tilde{u}^2}=\text{constant}\,. \tag{3.61b}$$

Thus, U must have the general form

$$U(\tilde{u},t|\tilde{x}_0,0)=\phi\big(\tilde{u}e^{-(\gamma/2)t}\big)e^{-(\bar{n}/4)\tilde{u}^2}, \tag{3.62}$$

where ϕ is an arbitrary function. Choosing ϕ to match the initial condition (3.59),

$$U(\tilde{u},t|\tilde{x}_0,0)=\exp\big[i\tilde{x}_0\tilde{u}e^{-(\gamma/2)t}\big]\exp\big[-(\bar{n}/4)\tilde{u}^2(1-e^{-\gamma t})\big]. \tag{3.63}$$

Taking the inverse Fourier transform, we have

$$\begin{aligned}
X(\tilde{x},t|\tilde{x}_0,0)&\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tilde{u}\,U(\tilde{u},t|\tilde{x}_0,0)e^{-i\tilde{x}\tilde{u}}\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tilde{u}\,\exp\big[-i\tilde{u}\big(\tilde{x}-\tilde{x}_0e^{-(\gamma/2)t}\big)\big]\\
&\quad\times\exp\big[-(\bar{n}/4)\tilde{u}^2(1-e^{-\gamma t})\big]\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tilde{u}\,\cos\big[\tilde{u}\big(\tilde{x}-\tilde{x}_0e^{-(\gamma/2)t}\big)\big]\exp\big[-(\bar{n}/4)\tilde{u}^2(1-e^{-\gamma t})\big]\\
&=\frac{1}{\sqrt{\pi\bar{n}(1-e^{-\gamma t})}}\exp\left[-\frac{\big(\tilde{x}-\tilde{x}_0e^{-(\gamma/2)t}\big)^2}{\bar{n}(1-e^{-\gamma t})}\right].
\end{aligned} \tag{3.64}$$

Equation (3.55b) can be solved in a similar fashion, whence,

$$\begin{aligned}
&\tilde{P}(\tilde{x},\tilde{y},t|\tilde{x}_0,\tilde{y}_0,0)\\
&\quad=\frac{1}{\pi\bar{n}(1-e^{-\gamma t})}\exp\left[-\frac{\big(\tilde{x}-\tilde{x}_0e^{-(\gamma/2)t}\big)^2+\big(\tilde{y}-\tilde{y}_0e^{-(\gamma/2)t}\big)^2}{\bar{n}(1-e^{-\gamma t})}\right],
\end{aligned} \tag{3.65}$$

or, equivalently,

$$\tilde{P}(\tilde{\alpha},\tilde{\alpha}^*,t|\tilde{\alpha}_0,\tilde{\alpha}_0^*,0)=\frac{1}{\pi\bar{n}(1-e^{-\gamma t})}\exp\left[-\frac{\big|\tilde{\alpha}-\tilde{\alpha}_0e^{-(\gamma/2)t}\big|^2}{\bar{n}(1-e^{-\gamma t})}\right]. \tag{3.66}$$

Then the P *distribution for a damped coherent state* is given by

$$P(\alpha,\alpha^*,t|\alpha_0,\alpha_0^*,0)=\frac{1}{\pi\bar{n}(1-e^{-\gamma t})}\exp\left[-\frac{\big|\alpha-\alpha_0e^{-(\gamma/2)t}e^{-i\omega_0t}\big|^2}{\bar{n}(1-e^{-\gamma t})}\right]. \tag{3.67}$$

$P(\alpha,\alpha^*,t|\alpha_0,\alpha_0^*,0)$ is a two-dimensional Gaussian distribution. Thus, for this example the P distribution has all the properties of a probability distribution. The mean of the Gaussian gives the oscillating and decaying oscillator amplitude calculated previously directly from the master equation [Eq. (1.78)]:

$$\langle a(t)\rangle=\big(\overline{\alpha(t)}\big)_P=\alpha_0e^{-(\gamma/2)t}e^{-i\omega_0t}. \tag{3.68}$$

The phase-independent variance describes the thermal fluctuations added to the coherent amplitude by the oscillator's interaction with the reservoir:

$$\begin{aligned}
\langle(a^\dagger a)(t)\rangle-\langle a^\dagger(t)\rangle\langle a(t)\rangle&=\big(\overline{(\alpha^*\alpha)(t)}\big)_P-\big(\overline{\alpha^*(t)}\big)_P\big(\overline{\alpha(t)}\big)_P\\
&=\Big[\big(\overline{x^2(t)}\big)_P+\big(\overline{y^2(t)}\big)_P\Big]-\Big[\big(\overline{x(t)}\big)_P^2+\big(\overline{y(t)}\big)_P^2\Big]\\
&=\bar{n}(1-e^{-\gamma t}).
\end{aligned} \tag{3.69}$$

For an initial coherent state, $\langle a^\dagger(t)\rangle\langle a(t)\rangle=|\alpha_0|^2e^{-\gamma t}=\langle(a^\dagger a)(0)\rangle e^{-\gamma t}$, and therefore (3.69) also agrees with our previous calculation [Eq. (1.80)]. In the long-time limit the coherent amplitude decays to zero and the variance of the fluctuations in each quadrature of the complex amplitude grows to $\bar{n}/2$. A comparison of (3.67) with (3.27) shows that the oscillator reaches a thermal state with mean photon number $\bar{n}$ equal to the mean photon number for a reservoir oscillator of frequency ω_0. Figure 3.1 illustrates these dynamics with $P(\alpha,\alpha^*,t|\alpha_0,\alpha_0^*,0)$ represented by a single circular contour of radius $\sqrt{(\bar{n}/2)(1-e^{-\gamma t})}$. For a Gaussian, the mean and variance determine all higher-order moments. Hence, (3.68) and (3.69) determine all of the *normal-ordered* operator averages for the damped oscillator [Eq. (3.19)]. Using the P representation we have put the statistical properties of the *quantum-mechanical* oscillator into a correspondence with a *classical* statistical description in terms of the phase-space variables x and y. (For a mechanical oscillator the coordinate and momentum variables are $q=x\sqrt{2\hbar/m\omega}$ and $p=y\sqrt{2\hbar m\omega}$, respectively.)

3.2 The Characteristic Function for Normal-Ordered Averages

We now look at an alternative way of defining the P representation and deriving an equation of motion for the P distribution. This second approach leaves the relationship to coherent states somewhat hidden, but introduces a method that can readily be generalized – to define representations based on

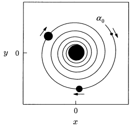

Fig. 3.1 Time evolution of $P(\alpha,\alpha^*,t|\,\alpha_0,\alpha_0^*,0)$ [Eq. (3.67)]. The center of the Gaussian distribution follows the spiral curve while the width of the distribution increases with time, as illustrated by the filled circular contours $c_t(\theta) = \alpha_0 e^{-(\gamma/2)t}\,e^{-i\omega_0 t} + e^{i\theta}\sqrt{(\bar{n}/2)(1-e^{-\gamma t})}$.

different operator orderings, and to define representations for collections of two-level atoms.

We have recently met two relationships that might suggest the new approach to us. In (3.23) and (3.24), and in Exercise 3.3, we saw that the Fourier transform of $P(\alpha,\alpha^*)$ played an important role. Why not begin from the function appearing on the left-hand side of (3.23) and *define* $P(\alpha,\alpha^*)$ to be its Fourier transform. Indeed, this approach is suggested on the following, more general grounds.

3.2.1 Operator Averages and the Characteristic Function

The function

$$\chi_N(z,z^*) \equiv \mathrm{tr}\big(\rho e^{iz^*a^\dagger}e^{iza}\big) \tag{3.70}$$

appearing on the left-hand side of (3.23) is a *characteristic function* in the usual sense of statistical physics [3.17]; it determines all *normal-ordered operator averages* via the prescription

$$\begin{aligned}\langle a^{\dagger p}a^q\rangle &\equiv \mathrm{tr}\big(\rho a^{\dagger p}a^q\big)\\ &= \frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\,\chi_N(z,z^*)\bigg|_{z=z^*=0}.\end{aligned} \tag{3.71}$$

The definition of a distribution for calculating normal-ordered averages follows quite naturally from this result. If we define $P(\alpha,\alpha^*)$ to be the two-dimensional Fourier transform of $\chi_N(z,z^*)$:

$$\begin{aligned}P(\alpha,\alpha^*) &\equiv \frac{1}{\pi^2}\int d^2z\,\chi_N(z,z^*)e^{-iz^*\alpha^*}e^{-iz\alpha}\\ &\equiv \frac{1}{\pi^2}\int_{-\infty}^{\infty}d\mu\int_{-\infty}^{\infty}d\nu\,\chi_N(\mu+i\nu,\mu-i\nu)e^{-2i(\mu x-\nu y)},\end{aligned} \tag{3.72}$$

with the inverse relationship

$$\begin{aligned}\chi_N(z,z^*) &= \int d^2\alpha\,P(\alpha,\alpha^*)e^{iz^*\alpha^*}e^{iz\alpha}\\ &= \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,P(x+iy,x-iy)e^{2i(\mu x-\nu y)},\end{aligned} \tag{3.73}$$

then, from (3.71) and (3.73),

$$\begin{aligned}\langle a^{\dagger p}a^q\rangle &= \frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\int d^2\alpha\,P(\alpha,\alpha^*)e^{iz^*\alpha^*}e^{iz\alpha}\bigg|_{z=z^*=0}\\ &= \big(\overline{\alpha^{*p}\alpha^q}\big)_P,\end{aligned} \tag{3.74a}$$

with

$$\big(\overline{\alpha^{*p}\alpha^q}\big)_P \equiv \int d^2\alpha\,P(\alpha,\alpha^*)\alpha^{*p}\alpha^q. \tag{3.74b}$$

Equation (3.73) is the same as (3.23), and (3.74) reproduces (3.19); the $P(\alpha,\alpha^*)$ defined in this way is the distribution introduced in (3.15) to give a diagonal expansion in terms of coherent states. Let us see how the Fokker–Planck equation for the damped harmonic oscillator can be derived by starting from this new definition of $P(\alpha,\alpha^*)$.

3.2.2 Derivation of the Fokker–Planck Equation Using the Characteristic Function

We will derive an equation of motion for the characteristic function and then use the relationship between $\chi_N(z,z^*,t)$ and $P(\alpha,\alpha^*,t)$ to convert this into an equation of motion for $P(\alpha,\alpha^*,t)$.

From the definition of χ_N,

$$\frac{\partial\chi_N}{\partial t} = \frac{\partial}{\partial t}\mathrm{tr}\big(\rho\,e^{iz^*a^\dagger}e^{iza}\big) = \mathrm{tr}\big(\dot\rho\,e^{iz^*a^\dagger}e^{iza}\big). \tag{3.75}$$

Then, the master equation (3.42) gives

$$\begin{aligned}\frac{\partial\chi_N}{\partial t} =\mathrm{tr}\Big\{\Big[&-i\omega_0(a^\dagger a\rho-\rho a^\dagger a)+\frac{\gamma}{2}(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a)\\ &+\gamma\bar{n}(a\rho a^\dagger+a^\dagger\rho a-a^\dagger a\rho-\rho a a^\dagger)\Big]e^{iz^*a^\dagger}e^{iza}\Big\}.\end{aligned} \tag{3.76}$$

Our aim is to express each of the nine terms on the right-hand side of (3.76) in terms of χ_N and its derivatives with respect to (iz^*) and (iz). For two of the nine terms this can be achieved directly; we may write

$$\begin{aligned}\mathrm{tr}\big(a\rho a^\dagger e^{iz^*a^\dagger}e^{iza}\big) &= \mathrm{tr}\big(\rho a^\dagger e^{iz^*a^\dagger}e^{iza}a\big)\\ &= \frac{\partial^2}{\partial(iz^*)\partial(iz)}\,\chi_N,\end{aligned} \tag{3.77}$$

where we have simply used the cyclic property of the trace. The remaining seven terms require a little more algebraic manipulation; but the goal is always the same – to rearrange the terms inside the trace so that $a^\dagger$ is to the left of $e^{iz^*a^\dagger}$ and a is to the right of e^{iza}. Then, $a^\dagger$ and a can be brought down from the exponentials by differentiation with respect to (iz^*) and (iz), respectively. Generally, the rearrangement may require us to pass $a^\dagger$ through the exponential e^{iza}, or a through the exponential $e^{iz^*a^\dagger}$. For this purpose we use

$$e^{iza}a^\dagger e^{-iza} = a^\dagger + iz, \tag{3.78a}$$
$$e^{-iz^*a^\dagger}a\,e^{iz^*a^\dagger} = a + iz^*. \tag{3.78b}$$

Equation (3.78a) follows by writing $a^\dagger(iz) = e^{iza}a^\dagger e^{-iza}$, with $a^\dagger(0) = a^\dagger$; then differentiate with respect to (iz):

$$\frac{d}{d(iz)}a^\dagger(iz) = e^{iza}(aa^\dagger - a^\dagger a)e^{-iza} = 1.$$

Thus,

$$a^\dagger(iz) = a^\dagger(0) + iz = a^\dagger + iz.$$

Equation (3.78b) is obtained as the Hermitian conjugate of (3.78a) and the replacement $z^* \to -z^*$.

Now, using (3.78) and the cyclic property of the trace, the remaining terms in (3.76) are:

$$\begin{aligned}\mathrm{tr}\big(a^\dagger a\rho e^{iz^*a^\dagger}e^{iza}\big) &= \mathrm{tr}\big(\rho e^{iz^*a^\dagger}e^{iza}a^\dagger a\big)\\ &= \mathrm{tr}\big[\rho e^{iz^*a^\dagger}\big(e^{iza}a^\dagger e^{-iza}\big)e^{iza}a\big]\\ &= \mathrm{tr}\big[\rho(a^\dagger + iz)e^{iz^*a^\dagger}e^{iza}a\big]\\ &= \left(\frac{\partial}{\partial(iz^*)} + iz\right)\mathrm{tr}\big(\rho\, e^{iz^*a^\dagger}e^{iza}a\big)\\ &= \left(\frac{\partial}{\partial(iz^*)} + iz\right)\frac{\partial}{\partial(iz)}\chi_N,\end{aligned} \tag{3.79}$$

$$\begin{aligned}\mathrm{tr}\big(\rho a^\dagger a\, e^{iz^*a^\dagger}e^{iza}\big) &= \mathrm{tr}\big[\rho a^\dagger e^{iz^*a^\dagger}\big(e^{-iz^*a^\dagger}ae^{iz^*a^\dagger}\big)e^{iza}\big]\\ &= \mathrm{tr}\big[\rho a^\dagger e^{iz^*a^\dagger}e^{iza}(a + iz^*)\big]\\ &= \left(\frac{\partial}{\partial(iz)} + iz^*\right)\mathrm{tr}\big(\rho a^\dagger e^{iz^*a^\dagger}e^{iza}\big)\\ &= \left(\frac{\partial}{\partial(iz)} + iz^*\right)\frac{\partial}{\partial(iz^*)}\chi_N,\end{aligned} \tag{3.80}$$

$$\begin{aligned}\mathrm{tr}\big(\rho aa^\dagger e^{iz^*a^\dagger}e^{iza}\big) &= \mathrm{tr}\big[\rho(a^\dagger a + 1)e^{iz^*a^\dagger}e^{iza}\big]\\ &= \left[\left(\frac{\partial}{\partial(iz)} + iz^*\right)\frac{\partial}{\partial(iz^*)} + 1\right]\chi_N,\end{aligned} \tag{3.81}$$

which follows from (3.80); the last term is left as an exercise:

Exercise 3.4 Show that

$$\begin{aligned}&\mathrm{tr}\big(a^\dagger\rho ae^{iz^*a^\dagger}e^{iza}\big)\\ &\quad = \left(1 - |z|^2 + iz\frac{\partial}{\partial(iz)} + iz^*\frac{\partial}{\partial(iz^*)} + \frac{\partial^2}{\partial(iz)\partial(iz^*)}\right)\chi_N.\end{aligned} \tag{3.82}$$

After substituting (3.77) and (3.79)–(3.82) into (3.76) the equation of motion for $\chi_N(z, z^*, t)$ is given by

$$\frac{\partial\chi_N}{\partial t} = \left[-\left(\frac{\gamma}{2} + i\omega_0\right)z\frac{\partial}{\partial z} - \left(\frac{\gamma}{2} - i\omega_0\right)z^*\frac{\partial}{\partial z^*} - \gamma\bar{n}zz^*\right]\chi_N. \tag{3.83}$$

To pass to an equation of motion for $P(\alpha, \alpha^*, t)$ we use the Fourier transform relation (3.73) and exchange the differential operator in the variables z and z^* for one in the variables α and α^*:

$$\begin{aligned}&\int d^2\alpha\,\frac{\partial P(\alpha,\alpha^*,t)}{\partial t}\,e^{iz^*\alpha^*}e^{iz\alpha}\\ &= \int d^2\alpha\,P(\alpha,\alpha^*,t)\bigg[-\left(\frac{\gamma}{2} + i\omega_0\right)z\frac{\partial}{\partial z} - \left(\frac{\gamma}{2} - i\omega_0\right)z^*\frac{\partial}{\partial z^*}\\ &\quad - \gamma\bar{n}zz^*\bigg]e^{iz^*\alpha^*}e^{iz\alpha}\\ &= \int d^2\alpha\,P(\alpha,\alpha^*,t)\bigg[-\left(\frac{\gamma}{2} + i\omega_0\right)(i\alpha)\frac{\partial}{\partial(i\alpha)} - \left(\frac{\gamma}{2} - i\omega_0\right)(i\alpha^*)\frac{\partial}{\partial(i\alpha^*)}\\ &\quad -\gamma\bar{n}\frac{\partial^2}{\partial(i\alpha)\partial(i\alpha^*)}\bigg]e^{iz^*\alpha^*}e^{iz\alpha}.\end{aligned} \tag{3.84}$$

The action of the derivatives on the right-hand side of (3.84) can be moved from the product of exponentials, $e^{iz^*\alpha^*}e^{iz\alpha}$, to $P(\alpha, \alpha^*, t)$ by integrating by parts; we took the same step in passing from (3.45) to (3.46). Once again we assume that $P(\alpha, \alpha^*, t)$ vanishes sufficiently fast at infinity to justify dropping the boundary terms. Then, (3.84) becomes

$$\begin{aligned}\int d^2\alpha\,e^{iz^*\alpha^*}e^{iz\alpha}\frac{\partial P}{\partial t} &= \int d^2\alpha\,e^{iz^*\alpha^*}e^{iz\alpha}\bigg[\left(\frac{\gamma}{2} + i\omega_0\right)\frac{\partial}{\partial\alpha}\alpha\\ &\quad + \left(\frac{\gamma}{2} - i\omega_0\right)\frac{\partial}{\partial\alpha^*}\alpha^* + \gamma\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\bigg]P.\end{aligned} \tag{3.85}$$

This is the Fourier transform of the Fokker–Planck equation derived in Sect. 3.1.4. It is precisely the equation derived from (3.46) in Exercise 3.3. Thus, after inverting the Fourier transform we arrive once again at the Fokker–Planck equation (3.47).

4. Quantum–Classical Correspondence for the Electromagnetic Field II: *P*, *Q*, and Wigner Representations

The definition of the P representation as the Fourier transform of the normal-ordered characteristic function can be generalized by simply taking different characteristic functions – characteristic functions that give operator averages in other than normal order. Here we will look at two new representations: the Q representation, which is defined in terms of the characteristic function that gives operator averages in antinormal order, and the Wigner representation, defined in terms of the characteristic function that gives operator averages in symmetric, or Weyl, order. This is not a comprehensive list. Cahill and Glauber [4.1], and Agarwal and Wolf [4.2] have introduced formalisms in which whole classes of different representations are defined. In particular, Agarwal and Wolf take the possibilities to their ultimate extreme and develop a very general and elegant formalism which they call the phase-space calculus. These general formalisms are not of much interest, however, when it comes to applications. The P, Q, and Wigner representations are the only examples that have traditionally seen any use in quantum optics. They are special cases within the classes defined by Cahill and Glauber, and Agarwal and Wolf. In Volume 2 we will meet one recent addition to the list which has been used quite extensively, particularly in the treatment of squeezing and related nonclassical effects. This is the positive P representation introduced by Drummond and Gardiner [4.3]. As the name suggests, the positive P representation is closely related to the Glauber–Sudarshan P representation. We postpone its discussion, however, until we have acquired the background needed to appreciate its special purpose and application. Certain properties of the positive P representation are still only partly understood; this representation therefore belongs with the modern research topics that are taken up in Volume 2.

For additional reading on the Q and Wigner representations reference may be made to Louisell [4.4] and Haken [4.5]. Also, Hillery *et al.* provide a comprehensive review with numerous references [4.6].

4.1 The Q and Wigner Representations

4.1.1 Antinormal-Ordered Averages and the Q Representation

If we wish to calculate antinormal-ordered averages, the rather obvious generalization from (3.70) is to define the characteristic function

$$\chi_A(z, z^*) \equiv \operatorname{tr}\big(\rho e^{iza} e^{iz^* a^\dagger}\big). \tag{4.1}$$

Then in place of (3.71), *antinormal-ordered operator averages* are given by

$$\begin{aligned}\langle a^q a^{\dagger p}\rangle &\equiv \operatorname{tr}\big(\rho a^q a^{\dagger p}\big)\\ &= \frac{\partial^{p+q}}{\partial(iz^*)^p \partial(iz)^q}\chi_A(z, z^*)\bigg|_{z=z^*=0}.\end{aligned} \tag{4.2}$$

If we define the distribution $Q(\alpha, \alpha^*)$ as the Fourier transform of $\chi_A(z, z^*)$:

$$\begin{aligned}Q(\alpha, \alpha^*) &\equiv \frac{1}{\pi^2}\int d^2z\, \chi_A(z, z^*) e^{-iz^*\alpha^*} e^{-iz\alpha}\\ &\equiv \frac{1}{\pi^2}\int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} d\nu\, \chi_A(\mu + i\nu, \mu - i\nu) e^{-2i(\mu x - \nu y)},\end{aligned} \tag{4.3}$$

with the inverse relationship

$$\begin{aligned}\chi_A(z, z^*) &= \int d^2\alpha\, Q(\alpha, \alpha^*) e^{iz^*\alpha^*} e^{iz\alpha}\\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, Q(x + iy, x - iy) e^{2i(\mu x - \nu y)},\end{aligned} \tag{4.4}$$

corresponding to (3.74), we now have

$$\begin{aligned}\langle a^q a^{\dagger p}\rangle &= \frac{\partial^{p+q}}{\partial(iz^*)^p \partial(iz)^q}\int d^2\alpha\, Q(\alpha, \alpha^*) e^{iz^*\alpha^*} e^{iz\alpha}\bigg|_{z=z^*=0}\\ &= \big(\overline{\alpha^{*p}\alpha^q}\big)_Q,\end{aligned} \tag{4.5a}$$

with

$$\big(\overline{\alpha^{*p}\alpha^q}\big)_Q \equiv \int d^2\alpha\, Q(\alpha, \alpha^*) \alpha^{*p}\alpha^q. \tag{4.5b}$$

The Q distribution, so defined, has a very simple relationship to the coherent states. Consider (4.3) with $\chi_A(z, z^*)$ substituted explicitly from (4.1) and the unit operator judiciously introduced in the form (3.9). We find

$$\begin{aligned}Q(\alpha, \alpha^*) &= \frac{1}{\pi^2}\int d^2z\, \operatorname{tr}\left[\rho e^{iza}\left(\frac{1}{\pi}\int d^2\lambda\, |\lambda\rangle\langle\lambda|\right) e^{iz^* a^\dagger}\right] e^{-iz^*\alpha^*} e^{-iz\alpha}\\ &= \frac{1}{\pi^3}\int d^2z \int d^2\lambda\, \langle\lambda| e^{iz^* a^\dagger} \rho e^{iza} |\lambda\rangle e^{-iz^*\alpha^*} e^{-iz\alpha}\\ &= \frac{1}{\pi}\int d^2\lambda\, \langle\lambda|\rho|\lambda\rangle \left[\frac{1}{\pi^2}\int d^2z\, e^{iz^*(\lambda^* - \alpha^*)} e^{iz(\lambda - \alpha)}\right]\end{aligned}$$

$$\begin{aligned}&= \frac{1}{\pi}\int d^2\lambda\, \langle\lambda|\rho|\lambda\rangle \delta^{(2)}(\lambda - \alpha)\\ &= \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle.\end{aligned} \tag{4.6}$$

Thus, $\pi Q(\alpha, \alpha^*)$ is the diagonal matrix element of the density operator taken with respect to the coherent state $|\alpha\rangle$. It is therefore strictly a probability – the probability for observing the coherent state $|\alpha\rangle$. This immediately gives us the relationship between Q and P.

From (3.21) and (4.6),

$$Q(\alpha, \alpha^*) = \frac{1}{\pi}\int d^2\lambda\, e^{-|\lambda - \alpha|^2} P(\lambda, \lambda^*). \tag{4.7}$$

Note 4.1 It can be shown that the diagonal matrix elements $\langle\alpha|\rho|\alpha\rangle$ specify the density operator completely. Then the convolution (4.7) forms the basis of formal proofs that every density operator may be given a diagonal representation if P is allowed to be a generalized function. See [4.7] and [4.8] for the details.

Another useful result is the relationship between the characteristic functions $\chi_A(z, z^*)$ and $\chi_N(z, z^*)$. We will make use of this shortly to derive the Fokker–Planck equation for the damped harmonic oscillator in the Q representation. The relationship follows from a special case of the Baker-Hausdorff theorem [4.9]: If $\hat{O}_1$ and $\hat{O}_2$ are two noncommuting operators that both commute with their commutator, then

$$e^{\hat{O}_1 + \hat{O}_2} = e^{\hat{O}_1} e^{\hat{O}_2} e^{-\frac{1}{2}[\hat{O}_1, \hat{O}_2]} = e^{\hat{O}_2} e^{\hat{O}_1} e^{\frac{1}{2}[\hat{O}_1, \hat{O}_2]}. \tag{4.8}$$

Since the commutator of a and $a^\dagger$ is a constant, this result can clearly be applied to the exponentials in the definitions of $\chi_N(z, z^*)$ and $\chi_A(z, z^*)$. It follows from (3.70) and (4.1) that

$$\begin{aligned}\chi_A(z, z^*) &\equiv \operatorname{tr}\big(\rho e^{iza} e^{iz^* a^\dagger}\big)\\ &= \operatorname{tr}\big(\rho e^{iza + iz^* a^\dagger}\big) e^{-\frac{1}{2}|z|^2}\\ &= \operatorname{tr}\big(\rho e^{iz^* a^\dagger} e^{iza}\big) e^{-|z|^2}\\ &= e^{-|z|^2}\chi_N(z, z^*).\end{aligned} \tag{4.9}$$

Exercise 4.1 Use (4.9) to derive (4.7) directly from the definitions of the Q and P distributions [Eqs. (4.3) and (3.72)]. Also, use both (3.40) and (3.41) to show that (4.7) gives the correct Q distribution for the Fock state $|l\rangle$ – namely;

$$\begin{aligned} Q(\alpha,\alpha^*) &= \frac{1}{\pi}|\langle\alpha|l\rangle|^2 \\ &= \frac{1}{\pi}e^{-|\alpha|^2}\frac{|\alpha|^{2l}}{l!}. \end{aligned} \tag{4.10}$$

An alternative relationship between the Q and P distributions follows from (4.9). Using (4.3) and (4.9),

$$\begin{aligned} Q(\alpha,\alpha^*) &= \frac{1}{\pi^2}\int d^2z\,\chi_A(z,z^*)e^{-iz^*\alpha^*}e^{-iz\alpha} \\ &= \frac{1}{\pi^2}\int d^2z\,e^{-|z|^2}\chi_N(z,z^*)e^{-iz^*\alpha^*}e^{-iz\alpha}. \end{aligned}$$

Then, writing $\chi_N(z,z^*)$ as the Fourier transform of $P(\lambda,\lambda^*)$, we have

$$\begin{aligned} &Q(\alpha,\alpha^*) \\ &= \frac{1}{\pi^2}\int d^2z\,e^{-|z|^2}\int d^2\lambda\,P(\lambda,\lambda^*)e^{iz^*\lambda^*}e^{iz\lambda}e^{-iz^*\alpha^*}e^{-iz\alpha} \\ &= \frac{1}{\pi^2}\int d^2z\int d^2\lambda\,P(\lambda,\lambda^*)\left[\exp\left(\frac{\partial^2}{\partial\lambda\partial\lambda^*}\right)e^{iz^*\lambda^*}e^{iz\lambda}\right]e^{-iz^*\alpha^*}e^{-iz\alpha} \\ &= \frac{1}{\pi^2}\int d^2z\int d^2\lambda\left[\exp\left(\frac{\partial^2}{\partial\lambda\partial\lambda^*}\right)P(\lambda,\lambda^*)\right]e^{iz^*(\lambda^*-\alpha^*)}e^{iz(\lambda-\alpha)}, \end{aligned}$$

where the last line follows after integrating by parts. The integral with respect to z gives a δ-function and we find

$$Q(\alpha,\alpha^*) = \exp\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)P(\alpha,\alpha^*). \tag{4.11}$$

Note 4.2 If (4.11) is to hold for the coherent state $|\alpha_0\rangle$, (4.7) and (3.22) require that we prove the rather unlikely looking result

$$\exp\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\delta^{(2)}(\alpha-\alpha_0) = \frac{1}{\pi}e^{-|\alpha-\alpha_0|^2}.$$

In spite of its unlikely appearance, this result follows from the limit defining the δ-function [Eq. (3.33)] and

$$\exp\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\frac{n}{\pi}e^{-n|\alpha|^2} = \frac{1}{\pi}\frac{n}{1+n}e^{-n|\alpha|^2/(1+n)}. \tag{4.12}$$

Equation (4.12) can be proved using the identity (4.46):

$$\begin{aligned} \exp\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)e^{-n|\alpha|^2} &= e^{-n|\alpha|^2}\sum_{k=0}^{\infty}\frac{(n|\alpha|)^{2k}}{k!}\frac{1}{(1+n)^{k+1}} \\ &= \frac{1}{1+n}e^{-n|\alpha|^2}e^{n^2|\alpha|^2/(1+n)} \\ &= \frac{1}{1+n}e^{-n|\alpha|^2/(1+n)}. \end{aligned}$$

4.1.2 The Damped Harmonic Oscillator in the Q Representation

A Fokker–Planck equation for the damped harmonic oscillator can be derived in the Q representation by following the same steps as in Sect. 3.2.2. A convenient shortcut is available, however; we can use the relationship (4.9) between $\chi_N(z,z^*)$ and $\chi_A(z,z^*)$ and the equation of motion (3.83) for χ_N to quickly arrive at the equation of motion for χ_A:

$$\begin{aligned} \frac{\partial\chi_A}{\partial t} &= e^{-|z|^2}\frac{\partial\chi_N}{\partial t} \\ &= e^{-|z|^2}\left[-\left(\frac{\gamma}{2}+i\omega_0\right)z\frac{\partial}{\partial z}-\left(\frac{\gamma}{2}-i\omega_0\right)z^*\frac{\partial}{\partial z^*}-\gamma\bar{n}zz^*\right]\chi_N \\ &= \left[-\left(\frac{\gamma}{2}+i\omega_0\right)z\left(\frac{\partial}{\partial z}+z^*\right)-\left(\frac{\gamma}{2}-i\omega_0\right)z^*\left(\frac{\partial}{\partial z^*}+z\right)\right. \\ &\qquad \left. -\gamma\bar{n}zz^*\right]e^{-|z|^2}\chi_N \\ &= \left[-\left(\frac{\gamma}{2}+i\omega_0\right)z\frac{\partial}{\partial z}-\left(\frac{\gamma}{2}-i\omega_0\right)z^*\frac{\partial}{\partial z^*}-\gamma(\bar{n}+1)zz^*\right]\chi_A. \end{aligned} \tag{4.13}$$

This is the same as the equation of motion for χ_N, except for the replacement $\bar{n}\to\bar{n}+1$. We can therefore write down the corresponding equation of motion for Q directly from (3.47):

$$\frac{\partial Q}{\partial t} = \left[\left(\frac{\gamma}{2}+i\omega_0\right)\frac{\partial}{\partial\alpha}\alpha+\left(\frac{\gamma}{2}-i\omega_0\right)\frac{\partial}{\partial\alpha^*}\alpha^*+\gamma(\bar{n}+1)\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]Q. \tag{4.14}$$

This is the *Fokker–Planck equation for the damped harmonic oscillator in the Q representation.*

We exploit the relationship between the Fokker–Planck equations in the P and Q representations further to solve (4.14). The Green function $Q(\alpha,\alpha^*,t|\alpha_0,\alpha_0^*,0)$, which has initial condition

$$Q(\alpha,\alpha^*,0|\alpha_0,\alpha_0^*,0) = \delta^{(2)}(\alpha-\alpha_0) \equiv \delta(x-x_0)\delta(y-y_0), \tag{4.15}$$

follows directly from (3.67) in the form

$$Q(\alpha,\alpha^*,t|\alpha_0,\alpha_0^*,0) = \frac{1}{\pi(\bar{n}+1)(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha-\alpha_0 e^{-(\gamma/2)t}e^{-i\omega_0 t}|^2}{(\bar{n}+1)(1-e^{-\gamma t})}\right]. \tag{4.16}$$

It is important to realize that while the Green function in the P representation describes an oscillator that is initially in a coherent state – $P(\alpha,\alpha^*,t|\alpha_0,\alpha_0^*,0) = P(\alpha,\alpha^*,t)_{\rho(0)=|\alpha_0\rangle\langle\alpha_0|}$ – the Green function in the Q representation does not describe an oscillator initially in a coherent state; a δ-function in the Q representation does not correspond to a coherent state. Indeed, (4.6) tells us that the Q distribution for an initial state $\rho(0)=|\alpha_0\rangle\langle\alpha_0|$ is

$$\begin{aligned} Q(\alpha,\alpha^*,0)_{\rho(0)=|\alpha_0\rangle\langle\alpha_0|} &= \frac{1}{\pi}\langle\alpha|\big(|\alpha_0\rangle\langle\alpha_0|\big)|\alpha\rangle \\ &= \frac{1}{\pi}|\langle\alpha|\alpha_0\rangle|^2 \\ &= \frac{1}{\pi}e^{-|\alpha-\alpha_0|^2}, \end{aligned} \tag{4.17}$$

where we have used (3.8). The time evolution of the Q distribution for this initial state is then calculated using

$$Q(\alpha,\alpha^*,t)_{\rho(0)=|\alpha_0\rangle\langle\alpha_0|} = \int d^2\lambda\, Q(\alpha,\alpha^*,t|\lambda,\lambda^*,0)Q(\lambda,\lambda^*,0)_{\rho(0)=|\alpha_0\rangle\langle\alpha_0|}. \tag{4.18}$$

Substituting (4.16) and (4.17) into (4.18), and making the change of variable $\lambda e^{-(\gamma/2)t}e^{-i\omega_0 t}\to\lambda$, we have

$$\begin{aligned} Q(\alpha,\alpha^*,t)_{\rho(0)=|\alpha_0\rangle\langle\alpha_0|} &= \int d^2\lambda\left\{\frac{1}{\pi(\bar{n}+1)(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha-\lambda e^{-(\gamma/2)t}e^{-i\omega_0 t}|^2}{(\bar{n}+1)(1-e^{-\gamma t})}\right]\right\} \\ &\quad\times\left\{\frac{1}{\pi}\exp\big[-|\lambda-\alpha_0|^2\big]\right\} \\ &= \int d^2\lambda\left\{\frac{1}{\pi(\bar{n}+1)(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha-\lambda|^2}{(\bar{n}+1)(1-e^{-\gamma t})}\right]\right\} \\ &\quad\times\left\{\frac{e^{\gamma t}}{\pi}\exp\Big[-|\lambda-\alpha_0 e^{-(\gamma/2)t}e^{-i\omega_0 t}|^2 e^{\gamma t}\Big]\right\}. \end{aligned} \tag{4.19}$$

This integral is a two-dimensional convolution; therefore, the Fourier transform of the left-hand side is given by the product of the Fourier transforms of the bracketed terms in the integrand; of course, the Fourier transform of the left-hand side is the characteristic function $\chi_A(z,z^*,t)_{\rho(0)=|\alpha_0\rangle\langle\alpha_0|}$. Thus,

$$\begin{aligned} \chi_A(z,z^*,t)_{\rho(0)=|\alpha_0\rangle\langle\alpha_0|} &= \exp\big[-|z|^2(\bar{n}+1)(1-e^{-\gamma t})\big] \\ &\quad\times\Big\{\exp\big[-|z|^2e^{-\gamma t}\big]e^{2iz^*\alpha_0^*(t)}e^{2iz\alpha_0(t)}\Big\}, \end{aligned} \tag{4.20}$$

with $\alpha_0(t)\equiv\alpha_0 e^{-(\gamma/2)t}e^{-i\omega_0 t}$. The inverse transform gives the Q *distribution for a damped coherent state*:

$$Q(\alpha,\alpha^*,t)_{\rho(0)=|\alpha_0\rangle\langle\alpha_0|} = \frac{1}{\pi[1+\bar{n}(1-e^{-\gamma t})]}\exp\left[-\frac{|\alpha-\alpha_0 e^{-(\gamma/2)t}e^{-i\omega_0 t}|^2}{1+\bar{n}(1-e^{-\gamma t})}\right]. \tag{4.21}$$

Compared with the solution for the P distribution [Eq. (3.67)], the solution (4.21) for the Q distribution shows one simple difference – the phase-independent variance [variance of $x\equiv\mathrm{Re}(\alpha)$ or $y\equiv\mathrm{Im}(\alpha)$] is now $(\bar{n}/2)(1-e^{-\gamma t})+1/2$ rather than $(\bar{n}/2)(1-e^{-\gamma t})$. Thus, the time evolution of the Q distribution can be represented as in Fig. 3.1, but with a circular contour of somewhat larger radius; in particular, the Q distribution has a width at $t=0$ given by the initial condition (4.17), whereas the P distribution begins as a δ-function; when $\bar{n}=0$, this initial width is preserved for all times. We find then that the Q distribution has a width even in the absence of thermal fluctuations. We have again set up a correspondence with a classical statistical process; but now there is noise where before there was none. What can this mean? The answer to this question illustrates an important point about the fluctuations at the "classical" end of the quantum–classical correspondence. Although thermal fluctuations from the reservoir are not too quantum mechanical – they should be present in a classical theory of damping also – in general, the fluctuations observed in the distributions derived via the quantum–classical correspondence have a quantum-mechanical origin. They are manifestations of the probabilistic character of quantum mechanics, and arise through the noncommutation of the quantum-mechanical operators. Therefore, the fluctuations that appear in the classical stochastic processes that correspond to a quantum-mechanical system via different operator orderings are different. In our present example, the difference in the variances of the P distribution and the Q distribution arises to preserve the boson commutation relation. From (3.74) and (3.67), we calculate

$$\begin{aligned} \langle(a^\dagger a)(t)\rangle - \langle a^\dagger(t)\rangle\langle a(t)\rangle &= \big(\overline{(\alpha^*\alpha)(t)}\big)_P - \big(\overline{\alpha^*(t)}\big)_P\big(\overline{\alpha(t)}\big)_P \\ &= \bar{n}(1-e^{-\gamma t}), \end{aligned} \tag{4.22a}$$

while from (4.5) and (4.21) we calculate

$$\begin{aligned} \langle(aa^\dagger)(t)\rangle - \langle a^\dagger(t)\rangle\langle a(t)\rangle &= \big(\overline{(\alpha^*\alpha)(t)}\big)_Q - \big(\overline{\alpha^*(t)}\big)_Q\big(\overline{\alpha(t)}\big)_Q \\ &= \bar{n}(1-e^{-\gamma t})+1. \end{aligned} \tag{4.22b}$$

The extra fluctuations in the Q representation, which give the "+1" in (4.22b), are just what are needed to preserve the expectation of the commutator – $\langle[a,a^\dagger](t)\rangle=1$.

4.1.3 Antinormal-Ordered Averages Using the P Representation

We should not be misled into thinking that the P and Q distributions are inadequate on their own for calculating operator averages in arbitrary order. Of course, an average in antinormal order can first be normal ordered so that moments of the P distribution can be used to calculate the average of the resulting normal-ordered object. Antinormal-ordered averages can also be evaluated, however, directly from the P distribution, without first reordering the operators. Consider (4.2) with $\chi_A(z,z^*)$ written in terms of $\chi_N(z,z^*)$ using (4.9). An arbitrary antinormal-average can be calculated from the relationship

$$\begin{aligned}\langle a^q a^{\dagger p}\rangle &= \frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}e^{-|z|^2}\chi_N(z,z^*)\Big|_{z=z^*=0}\\ &= \frac{\partial^p}{\partial(iz^*)^p}e^{-|z|^2}\left(iz^*+\frac{\partial}{\partial(iz)}\right)^q\chi_N(z,z^*)\Big|_{z=z^*=0}\\ &= e^{-|z|^2}\left(iz+\frac{\partial}{\partial(iz^*)}\right)^p\left(iz^*+\frac{\partial}{\partial(iz)}\right)^q\chi_N(z,z^*)\Big|_{z=z^*=0}\\ &= \frac{\partial^p}{\partial(iz^*)^p}\left(iz^*+\frac{\partial}{\partial(iz)}\right)^q\chi_N(z,z^*)\Big|_{z=z^*=0}.\end{aligned}$$

Substituting for $\chi_N(z,z^*)$ from (3.73), we have

$$\begin{aligned}\langle a^q a^{\dagger p}\rangle &= \int d^2\alpha\, P(\alpha,\alpha^*)\frac{\partial^p}{\partial(iz^*)^p}\left(iz^*+\frac{\partial}{\partial(iz)}\right)^q e^{iz^*\alpha^*}e^{iz\alpha}\Big|_{z=z^*=0}\\ &= \int d^2\alpha\, P(\alpha,\alpha^*)\left(\frac{\partial}{\partial\alpha^*}+\alpha\right)^q\alpha^{*p}e^{iz^*\alpha^*}e^{iz\alpha}\Big|_{z=z^*=0}.\end{aligned}$$

We now integrate by parts, setting $P(z,z^*)$ and its derivatives to zero at infinity, to arrive at the result

$$\langle a^q a^{\dagger p}\rangle = \int d^2\alpha\,\alpha^{*p}\left(\alpha-\frac{\partial}{\partial\alpha^*}\right)^q P(\alpha,\alpha^*). \tag{4.23a}$$

Exercise 4.2 Prove also that

$$\langle a^q a^{\dagger p}\rangle = \int d^2\alpha\,\alpha^{q}\left(\alpha^*-\frac{\partial}{\partial\alpha}\right)^p P(\alpha,\alpha^*), \tag{4.23b}$$

and

$$\langle a^{\dagger p} a^{q}\rangle = \int d^2\alpha\,\alpha^{*p}\left(\alpha+\frac{\partial}{\partial\alpha^*}\right)^q Q(\alpha,\alpha^*), \tag{4.24a}$$

$$\langle a^{\dagger p} a^{q}\rangle = \int d^2\alpha\,\alpha^{q}\left(\alpha^*+\frac{\partial}{\partial\alpha}\right)^p Q(\alpha,\alpha^*). \tag{4.24b}$$

As an illustration, let us calculate $\langle(aa^\dagger)(t)\rangle$ for the damped harmonic oscillator using (4.23a) and the Green function solution for the P distribution [Eq. (3.67)]. We set $\alpha_0(t) equive^{-(\gamma/2)t}e^{-i\omega_0 t}$ and then

$$\begin{aligned}&\langle(aa^\dagger)(t)\rangle\\ &= \int d^2\alpha\,\alpha^*\left(\alpha-\frac{\partial}{\partial\alpha^*}\right)\left\{\frac{1}{\pi\bar n(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha-\alpha_0(t)|^2}{\bar n(1-e^{-\gamma t})}\right]\right\}\\ &= \int d^2\alpha\,\alpha^*\left[\alpha+\frac{\alpha-\alpha_0(t)}{\bar n(1-e^{-\gamma t})}\right]\left\{\frac{1}{\pi\bar n(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha-\alpha_0(t)|^2}{\bar n(1-e^{-\gamma t})}\right]\right\}\\ &= \int d^2\alpha\left\{\alpha^*[\alpha-\alpha_0(t)]\left[1+\frac{1}{\bar n(1-e^{-\gamma t})}\right]+\alpha^*\alpha_0(t)\right\}\\ &\quad\times\left\{\frac{1}{\pi\bar n(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha-\alpha_0(t)|^2}{\bar n(1-e^{-\gamma t})}\right]\right\}.\end{aligned}$$

If A is a constant,

$$\begin{aligned}\int d^2\alpha\,\alpha\frac{A}{\pi}\exp\left[-A|\alpha-\alpha_0(t)|^2\right] &= \alpha_0(t),\\ \int d^2\alpha\,|\alpha-\alpha_0(t)|^2\frac{A}{\pi}\exp\left[-A|\alpha-\alpha_0(t)|^2\right] &= \frac{1}{A}.\end{aligned}$$

We can therefore replace α^* by $\alpha^*-\alpha_0^*(t)$ in the first term in the integrand (this adds zero to the integral) and perform the resulting integrals to obtain

$$\begin{aligned}\langle(aa^\dagger)(t)\rangle &= \int d^2\alpha\left[|\alpha-\alpha_0(t)|^2\frac{\bar n(1-e^{-\gamma t})+1}{\bar n(1-e^{-\gamma t})}+\alpha^*\alpha_0(t)\right]\\ &\quad\times\left\{\frac{1}{\pi\bar n(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha-\alpha_0(t)|^2}{\bar n(1-e^{-\gamma t})}\right]\right\}\\ &= \bar n(1-e^{-\gamma t})+1+|\alpha_0(t)|^2\\ &= \langle(a^\dagger a)(t)\rangle+1,\end{aligned}$$

where the last line follows from (3.68) and (3.69). We have arrived at the result that would be obtained by first writing $aa^\dagger$ in normal order and then using moments of the P distribution to evaluate the normal-ordered operator average.

4.1.4 The Wigner Representation

The Wigner representation is introduced by defining a third characteristic function:

$$\chi_S(z,z^*) \equiv \mathrm{tr}\big(\rho e^{iz^*a^\dagger + iza}\big). \tag{4.25}$$

The Wigner distribution $W(\alpha,\alpha^*)$ is the Fourier transform of $\chi_S(z,z^*)$:

$$\begin{aligned} W(\alpha,\alpha^*) &\equiv \frac{1}{\pi^2}\int d^2z\,\chi_S(z,z^*)e^{-iz^*\alpha^*}e^{-iz\alpha} \\ &\equiv \frac{1}{\pi^2}\int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} d\nu\,\chi_S(\mu+i\nu,\mu-i\nu)e^{-2i(\mu x-\nu y)}, \end{aligned} \tag{4.26}$$

with the inverse relationship

$$\begin{aligned} \chi_S(z,z^*) &= \int d^2\alpha\,W(\alpha,\alpha^*)e^{iz^*\alpha^*}e^{iz\alpha} \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\,W(x+iy,x-iy)e^{2i(\mu x-\nu y)}. \end{aligned} \tag{4.27}$$

The relationship between the Wigner distribution and operator averages is a little more complicated than the relationships that connect the P and Q distributions with operator averages. In terms of position and momentum variables (proportional to x and y respectively) the moments of $W(\alpha,\alpha^*)$ give the averages of operators written in Weyl order [4.10]. Details can be found in the review by Hillery *et al.* [4.6]. The relevant quantities for quantum optics are operator averages corresponding to moments of the complex variables α and α^*. These can be found as follows. The exponential in (4.25) has the expansion

$$\begin{aligned} e^{iz^*a^\dagger+iza} &= \sum_{m=0}^{\infty}\frac{1}{m!}(iz^*a^\dagger+iza)^m \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}(iz^*)^n(iz)^{m-n}\big(a^{\dagger n}a^{m-n}\big)_S \\ &= \sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{(iz^*)^n(iz)^{m-n}}{n!(m-n)!}\big(a^{\dagger n}a^{m-n}\big)_S \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(iz^*)^n(iz)^m}{n!m!}\big(a^{\dagger n}a^m\big)_S, \end{aligned} \tag{4.28}$$

where $\big(a^{\dagger n}a^m\big)_S$ denotes the operator product written in symmetric order – the average of $(n+m)!/(n!m!)$ possible orderings of n creation operators and m annihilation operators:

$$\begin{aligned} \big(a^\dagger a\big)_S &\equiv \tfrac{1}{2}(a^\dagger a + aa^\dagger), && (4.29a) \\ \big(a^{\dagger 2}a\big)_S &\equiv \tfrac{1}{3}(a^{\dagger 2}a + a^\dagger a a^\dagger + aa^{\dagger 2}), && (4.29b) \\ \big(a^\dagger a^2\big)_S &\equiv \tfrac{1}{3}(a^\dagger a^2 + aa^\dagger a + a^2a^\dagger), && (4.29c) \\ \big(a^{\dagger 2}a^2\big)_S &\equiv \tfrac{1}{6}(a^{\dagger 2}a^2 + a^\dagger a a^\dagger a + a^\dagger a^2 a^\dagger + aa^{\dagger 2}a + aa^\dagger a a^\dagger + a^2a^{\dagger 2}), && \\ &\;\;\vdots && (4.29d) \end{aligned}$$

Then, from (4.28) and the definition of $\chi_S(z,z^*)$ [Eq. (4.25)], *symmetric-ordered operator averages* are given by

$$\begin{aligned} \langle\big(a^{\dagger p}a^q\big)_S\rangle &\equiv \mathrm{tr}\big[\rho\big(a^{\dagger p}a^q\big)_S\big] \\ &= \frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\chi_S(z^*,z)\bigg|_{z=z^*=0}; \end{aligned} \tag{4.30}$$

substituting for $\chi_S(z,z^*)$ in terms of $W(\alpha,\alpha^*)$ [Eq. (4.27)] gives

$$\begin{aligned} \langle\big(a^{\dagger p}a^q\big)_S\rangle &= \frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\int d^2\alpha\,W(\alpha,\alpha^*)e^{iz^*\alpha^*}e^{iz\alpha}\bigg|_{z=z^*=0} \\ &= \big(\overline{\alpha^{*p}\alpha^q}\big)_W, \end{aligned} \tag{4.31a}$$

with

$$\big(\overline{\alpha^{*p}\alpha^q}\big)_W \equiv \int d^2\alpha\,W(\alpha,\alpha^*)\alpha^{*p}\alpha^q. \tag{4.31b}$$

Note 4.3 We have defined the Wigner distribution $W(\alpha,\alpha^*)$ to be normalized such that $\int d^2\alpha\,W(\alpha,\alpha^*)=1$. The Wigner distribution is often defined with a different normalization, such that $\int d^2\alpha\,W(\alpha,\alpha^*)=\pi$. This is the case in [4.4] and [4.6]. With the alternative definition $W(\alpha,\alpha^*)$ is the classical function associated with the density operator ρ by writing it as a power series in symmetric-ordered operators $\big(a^{\dagger p}a^q\big)_S$ and replacing each term in this series by $\alpha^{*p}\alpha^q$ (see Sect. 4.3.1).

The quantum–classical correspondence defined in terms of symmetric-ordered operators (also antinormal-ordered operators) is not really the most convenient for applications in quantum optics because it is normal-ordered averages that relate directly to quantities measured with detectors that absorb photons. However, often only low-order moments are of interest and the symmetric ordering is then easily untangled using (4.29a)–(4.29d). More generally, a symmetric-ordered operator can be written in normal order in the following way. With the help of the Baker-Hausdorff theorem [Eq. (4.8)] we write

$$\begin{aligned}(a^{\dagger p}a^q)_S &= \frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}e^{iz^*a^\dagger+iza}\bigg|_{z=z^*=0}\\ &= \frac{\partial^{p+q}}{\partial(iz^*)^p(iz)^q}e^{-\frac{1}{2}|z|^2}e^{iz^*a^\dagger}e^{iza}\bigg|_{z=z^*=0}.\end{aligned}$$

It can then be proved by induction that

$$\begin{aligned}&\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}e^{-\frac{1}{2}|z|^2}e^{iz^*a^\dagger}e^{iza}\\ &\quad= \sum_{k=0}^{\min(p,q)}\frac{1}{2^k}\frac{1}{k!}\frac{p!}{(p-k)!}\frac{q!}{(q-k)!}\Big(a^\dagger+\frac{1}{2}iz\Big)^{p-k}\\ &\qquad\times e^{-\frac{1}{2}|z|^2}e^{iz^*a^\dagger}e^{iza}\Big(a+\frac{1}{2}iz^*\Big)^{q-k},\end{aligned}\tag{4.32}$$

and hence, that

$$(a^{\dagger p}a^q)_S = \sum_{k=0}^{\min(p,q)}\frac{1}{2^k}\frac{p!}{(p-k)!}\frac{q!}{(q-k)!}a^{\dagger p-k}a^{q-k}.\tag{4.33}$$

The Baker-Hausdorff theorem also yields the relationship between the characteristic functions $\chi_S(z,z^*)$ and $\chi_N(z,z^*)$, and $\chi_S(z,z^*)$ and $\chi_A(z,z^*)$:

$$\chi_S(z,z^*) \equiv \mathrm{tr}\Big(\rho e^{iz^*a^\dagger+iza}\Big) = \mathrm{tr}\Big(\rho e^{iz^*a^\dagger}e^{iza}\Big)e^{-\frac{1}{2}|z|^2} = e^{-\frac{1}{2}|z|^2}\chi_N(z,z^*),\tag{4.34a}$$

$$\chi_S(z,z^*) \equiv \mathrm{tr}\Big(\rho e^{iz^*a^\dagger+iza}\Big) = \mathrm{tr}\Big(\rho e^{iza}e^{iz^*a^\dagger}\Big)e^{\frac{1}{2}|z|^2} = e^{\frac{1}{2}|z|^2}\chi_A(z,z^*).\tag{4.34b}$$

From these results relationships between the distributions $W(\alpha,\alpha^*)$ and $P(\alpha,\alpha^*)$, and $W(\alpha,\alpha^*)$ and $Q(\alpha,\alpha^*)$, analogous to those given in (4.7) and (4.11), can be obtained. The derivations are left as an exercise:

Exercise 4.3 Show that

$$W(\alpha,\alpha^*) = \frac{2}{\pi}\int d^2\lambda\; e^{-2|\lambda-\alpha|^2}P(\lambda,\lambda^*),\tag{4.35a}$$

$$Q(\alpha,\alpha^*) = \frac{2}{\pi}\int d^2\lambda\, e^{-2|\lambda-\alpha|^2}W(\lambda,\lambda^*),\tag{4.35b}$$

and that

$$W(\alpha,\alpha^*) = \exp\Big(\frac{1}{2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\Big)P(\alpha,\alpha^*),\tag{4.36a}$$

$$Q(\alpha,\alpha^*) = \exp\Big(\frac{1}{2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\Big)W(\alpha,\alpha^*).\tag{4.36b}$$

From the relationships (4.7) and (4.35), (4.9) and (4.34), and (4.11) and (4.6), the Wigner distribution appears to fall in some sense in between the P and Q distributions. This observation is illustrated explicitly by the example of the damped harmonic oscillator. There is no need for a new calculation to treat this example in the Wigner representation. From a comparison of (4.9) and (4.34a), we immediately conclude that the method of Sect. 4.1.2 will bring us to the following *Fokker–Planck equation for the damped harmonic oscillator in the Wigner representation*:

$$\frac{\partial W}{\partial t} = \bigg[\Big(\frac{\gamma}{2}+i\omega_0\Big)\frac{\partial}{\partial\alpha}\alpha + \Big(\frac{\gamma}{2}-i\omega_0\Big)\frac{\partial}{\partial\alpha^*}\alpha^* + \gamma(\bar{n}+\tfrac{1}{2})\frac{\partial^2}{\partial\alpha\partial\alpha^*}\bigg]W.\tag{4.37}$$

Thus, where $\bar{n}$ appears in the Fokker–Planck equation in the P representation [Eq. (3.47)], and $\bar{n}+1$ appears in the Fokker–Planck equation in the Q representation [Eq. (4.14)], now $\bar{n}+\frac{1}{2}$ appears in the Fokker–Planck equation in the Wigner representation. The factor of $\frac{1}{2}$ carries over into the solution for a damped coherent state. By referring to (3.67) and (4.16) we see that the Green function $W(\alpha,\alpha^*,t|\alpha_0,\alpha_0^*,0)$, which has initial condition

$$W(\alpha,\alpha^*,0|\alpha_0,\alpha_0^*,0) = \delta^{(2)}(\alpha-\alpha_0) \equiv \delta(x-x_0)\delta(y-y_0),\tag{4.38}$$

is given by

$$\begin{aligned}&W(\alpha,\alpha^*,t|\alpha_0,\alpha_0^*,0)\\ &\quad= \frac{1}{\pi(\bar{n}+\frac{1}{2})(1-e^{-\gamma t})}\exp\bigg[-\frac{|\alpha-\alpha_0e^{-(\gamma/2)t}e^{-i\omega_0t}|^2}{(\bar{n}+\frac{1}{2})(1-e^{-\gamma t})}\bigg].\end{aligned}\tag{4.39}$$

Then, using (4.35a) and the P distribution for a coherent state [Eq. (3.22)], an initial coherent state ($\rho(0)=|\alpha_0\rangle\langle\alpha_0|$) is represented by the distribution

$$W(\alpha,\alpha^*,0)_{\rho(0)=|\alpha_0\rangle\langle\alpha_0|} = \frac{2}{\pi}e^{-2|\alpha-\alpha_0|^2}.\tag{4.40}$$

By following the steps used to derive (4.21) we find that the *Wigner distribution for a damped coherent state* is given by

$$\begin{aligned}&W(\alpha,\alpha^*,t)_{\rho(0)=|\alpha_0\rangle\langle\alpha_0|}\\ &\quad= \frac{1}{\pi\big[\frac{1}{2}+\bar{n}(1-e^{-\gamma t})\big]}\exp\bigg[-\frac{|\alpha-\alpha_0e^{-(\gamma/2)t}e^{-i\omega_0t}|^2}{\frac{1}{2}+\bar{n}(1-e^{-\gamma t})}\bigg].\end{aligned}\tag{4.41}$$

We have now constructed a third correspondence with a classical statistical process. Here the phase-independent variance lies in between those given by the solutions (3.67) and (4.21); the picture of Fig. 3.1 still applies, but now with a circular contour of radius $\sqrt{1/2+\bar{n}(1-e^{-\gamma t})}$ representing the distribution. As we observed for the Q distribution, the quantum fluctuations added over and above those coming from the reservoir are required by the commutation relations and the ordering convention underlying the representation. From (4.29a), (4.31), and (4.41), we have

$$\begin{aligned}
&\tfrac{1}{2}\left[\langle(a^\dagger a)(t)\rangle+\langle(aa^\dagger)(t)\rangle\right]-\langle a^\dagger(t)\rangle\langle a(t)\rangle\\
&\qquad=\langle(a^\dagger a)_S(t)\rangle-\langle(a^\dagger)_S(t)\rangle\langle(a)_S(t)\rangle\\
&\qquad=\left(\overline{(\alpha^*\alpha)(t)}\right)_W-\left(\overline{\alpha^*(t)}\right)_W\left(\overline{\alpha(t)}\right)_W\\
&\qquad=\bar{n}(1-e^{-\gamma t})+\tfrac{1}{2}. \qquad (4.42)
\end{aligned}$$

This is the average of the expressions in (4.22a) and (4.22b). The factor "$+\frac{1}{2}$" is the contribution obtained from the boson commutation relation by normal ordering the operator $(a^\dagger a)_S=\frac{1}{2}(a^\dagger a+aa^\dagger)$.

4.2 Fun with Fock States

We have followed the treatment of the damped harmonic oscillator prepared in a coherent state throughout our discussions of the P, Q, and Wigner representations. For this example, each of the three distributions has all the properties of a probability distribution, and we can therefore associate the quantum-mechanical problem with each of three classical statistical descriptions. We should remember, however, that the distributions obtained from the quantum–classical correspondence are not guaranteed to have all the properties of a probability distribution. We have already seen in Sect. 3.1.3 that the P distribution for a Fock state is a generalized function, involving derivatives of the δ-function. We now explore the representation of Fock states a little further.

4.2.1 Wigner Distribution for a Fock State

Let us derive the Wigner distribution for the Fock state $|l\rangle$ using (4.35a) and the form of the P distribution given in (3.40). We have

$$\begin{aligned}
W(\alpha,\alpha^*)&=\frac{2}{\pi}\int d^2\lambda\, e^{-2|\lambda-\alpha|^2}\frac{1}{l!}e^{|\lambda|^2}\frac{\partial^{2l}}{\partial\lambda^l\partial\lambda^{*l}}\delta^{(2)}(\lambda)\\
&=\frac{2}{\pi}\frac{1}{l!}\frac{\partial^{2l}}{\partial\lambda^l\partial\lambda^{*l}}e^{-2|\lambda-\alpha|^2}e^{|\lambda|^2}\bigg|_{\lambda=\lambda^*=0}\\
&=\frac{2}{\pi}\frac{1}{l!}e^{-2|\alpha|^2}\frac{\partial^{2l}}{\partial\lambda^l\partial\lambda^{*l}}e^{-2|\lambda|^2}e^{2\lambda\alpha^*}e^{2\lambda^*\alpha}\bigg|_{\lambda=\lambda^*=0}. \qquad (4.43)
\end{aligned}$$

To evaluate the right-hand side of (4.43) we consider the more general expression (for any complex constants A, B, and C)

$$\begin{aligned}
\frac{\partial^{2l}}{\partial\lambda^l\partial\lambda^{*l}}e^{-A|\lambda|^2}e^{B\lambda}e^{C\lambda^*}&=\frac{\partial^l}{\partial\lambda^l}e^{B\lambda}\frac{\partial^l}{\partial\lambda^{*l}}e^{C\lambda^*}e^{-A|\lambda|^2}\\
&=e^{B\lambda}e^{C\lambda^*}\left(B+\frac{\partial}{\partial\lambda}\right)^l\left(C+\frac{\partial}{\partial\lambda^*}\right)^l e^{-A|\lambda|^2}\\
&=e^{B\lambda}e^{C\lambda^*}\left(B+\frac{\partial}{\partial\lambda}\right)^l(C-A\lambda)^l e^{-A|\lambda|^2}. \qquad (4.44)
\end{aligned}$$

For $n\le l$, it can be proved by induction that

$$\begin{aligned}
&\left(B+\frac{\partial}{\partial\lambda}\right)^l(C-A\lambda)^l\\
&=\left(B+\frac{\partial}{\partial\lambda}\right)^{l-n}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{l!}{(l-k)!}A^k(C-A\lambda)^{l-k}\left(B+\frac{\partial}{\partial\lambda}\right)^{n-k}. \qquad (4.45)
\end{aligned}$$

Using this result, with $n=l$, we obtain

$$\begin{aligned}
&\frac{\partial^{2l}}{\partial\lambda^l\partial\lambda^{*l}}e^{-A|\lambda|^2}e^{B\lambda}e^{C\lambda^*}\\
&=e^{B\lambda}e^{C\lambda^*}\sum_{k=0}^{l}(-1)^k\frac{l!}{k!(l-k)!}\frac{l!}{(l-k)!}A^k(C-A\lambda)^{l-k}\\
&\quad\times\left(B+\frac{\partial}{\partial\lambda}\right)^{l-k}e^{-A|\lambda|^2}\\
&=e^{-A|\lambda|^2}e^{B\lambda}e^{C\lambda^*}\sum_{k=0}^{l}(-1)^{l-k}\frac{l!}{k!(l-k)!}\frac{l!}{k!}A^{l-k}(B-A\lambda^*)^k(C-A\lambda)^k, \qquad (4.46)
\end{aligned}$$

where in the last line we have changed the summation index, with $l-k\to k$. The right-hand side of (4.43) may now be evaluated using (4.46): setting $A=1$ and $B^*=C=2\alpha$, the Wigner distribution for the Fock state $|l\rangle$ is given by

$$W(\alpha,\alpha^*)=\frac{2}{\pi}\frac{1}{l!}e^{-2|\alpha|^2}\sum_{k=0}^{l}(-1)^{l-k}\frac{l!}{k!(l-k)!}\frac{l!}{k!}|2\alpha|^{2k}. \qquad (4.47)$$

The distribution (4.47) is an ordinary, well-behaved, function. Nevertheless, it can clearly violate one of the conditions required of a probability distribution – it need not be positive. The one-photon Fock state illustrates this point; for $l=1$,

$$W(\alpha,\alpha^*) = \frac{2}{\pi} e^{-2|\alpha|^2}(4|\alpha|^2 - 1), \tag{4.48}$$

which is negative for $|\alpha| < \frac{1}{2}$.

Note 4.4 It can be shown that $\chi_S(z,z^*)$ is square integrable and, hence, that its Fourier transform $W(\alpha,\alpha^*)$ is *always* a well-behaved function; there is no need for generalized functions in the Wigner representation. To prove this result we use (4.34) and (4.1) to write

$$\begin{aligned}\frac{1}{\pi}\int d^2z\,|\chi_S(z,z^*)|^2 &= \frac{1}{\pi}\int d^2z\,\chi_N(z,z^*)\chi_A(z,z^*)^* \\ &= \frac{1}{\pi}\mathrm{tr}\left[\int d^2z\,\chi_N(z,z^*)e^{-iz^*a^\dagger}\rho e^{-iza}\right].\end{aligned}$$

Then, introducing the identity in the form (3.9) and using the cyclic property of the trace, and the relationship between $\chi_N(z,z^*)$ and $P(\alpha,\alpha^*)$ [Eq. (3.72)], we find

$$\begin{aligned}\frac{1}{\pi}\int d^2z\,|\chi_S(z,z^*)|^2 &= \frac{1}{\pi^2}\mathrm{tr}\left[\int d^2\alpha\int d^2z\,\chi_N(z,z^*)\langle\alpha|e^{-iz^*a^\dagger}\rho e^{-iza}|\alpha\rangle\right] \\ &= \frac{1}{\pi^2}\mathrm{tr}\left[\int d^2\alpha\,\langle\alpha|\rho|\alpha\rangle\int d^2z\,\chi_N(z,z^*)e^{-iz^*\alpha^*}e^{-iz\alpha}\right] \\ &= \mathrm{tr}\left[\rho\int d^2\alpha\,|\alpha\rangle\langle\alpha|P(\alpha,\alpha^*)\right] \\ &= \mathrm{tr}\,(\rho^2).\end{aligned}$$

The last line follows from (3.15). The square integrability of $\chi_S(z,z^*)$ follows because $\mathrm{tr}\,(\rho^2)\le 1$.

As a simple check on our result for the Fock state Wigner distribution, let us evaluate $(\overline{\alpha^*\alpha})_W$ and show that it gives the symmetric-ordered average

$$\tfrac{1}{2}\langle a^\dagger a + aa^\dagger\rangle = \tfrac{1}{2}\big(2\langle a^\dagger a\rangle + 1\big) = \tfrac{1}{2}(2l+1). \tag{4.49}$$

From (4.47) we obtain

$$\begin{aligned}(\overline{\alpha^*\alpha})_W &\equiv \int d^2\alpha\,W(\alpha,\alpha^*)\alpha^*\alpha \\ &= \frac{2}{\pi}\frac{1}{l!}\sum_{k=0}^{l}(-1)^{l-k}\frac{l!}{k!(l-k)!}\frac{l!}{k!}\int d^2\alpha\,e^{-2|\alpha|^2}|2\alpha|^{2k}|\alpha|^2 \\ &= \frac{2}{\pi}\sum_{k=0}^{l}(-1)^{l-k}\frac{l!}{(k!)^2(l-k)!}2^{2k}\int_0^\infty dr\int_0^{2\pi}d\phi\,e^{-2r^2}r^{2(k+1)+1} \\ &= \frac{2}{\pi}\sum_{k=0}^{l}(-1)^{l-k}\frac{l!}{(k!)^2(l-k)!}2^{2k}2\pi\frac{(k+1)!}{2^{k+3}}.\end{aligned}$$

The integral over r has been executed by performing $k+1$ integrations by parts. The summation on the right-hand side may now be split into two pieces by writing

$$\sum_{k=0}^{l}\cdots\frac{(k+1)!}{(k!)^2} = \sum_{k=1}^{l}\cdots\frac{1}{(k-1)!} + \sum_{k=0}^{l}\cdots\frac{1}{k!}.$$

Then, changing the first summation index, with $k-1\to k$, we arrive at the result

$$\begin{aligned}(\overline{\alpha^*\alpha})_W &= \frac{1}{2}\left[2l\sum_{k=0}^{l-1}(-1)^{(l-1)-k}\frac{(l-1)!}{k![(l-1)-k]!}2^k + \sum_{k=0}^{l}(-1)^{l-k}\frac{l!}{k!(l-k)!}2^k\right] \\ &= \tfrac{1}{2}\left[2l(2-1)^{l-1} + (2-1)^l\right] \\ &= \tfrac{1}{2}(2l+1).\end{aligned}$$

Thus, we recover the symmetric-ordered operator average (4.49) for a Fock state.

4.2.2 Damped Fock State in the P Representation

Nothing in the derivation of the Fokker–Planck equation for the damped harmonic oscillator precludes its use in situations where the distribution is a generalized function, or takes negative values. We certainly lose the correspondence with a classical statistical description under such circumstances, but the mathematics works just fine. The Green function for the appropriate Fokker–Planck equation provides all we need to find the time evolution from an arbitrary initial state; we simply integrate the Green function against the representation for the initial state. This will work even if the initial state is represented by a distribution that is more singular than a δ-function. For an interesting illustration we will calculate the P distribution for a damped harmonic oscillator prepared in the Fock state $|l\rangle$. Recall that a Fock state is represented by a distribution involving derivatives of a two-dimensional δ-function.

The Green function solution to the Fokker–Planck equation in the P representation is given by (3.67). Using this result and the distribution for an initial Fock state [Eq. (3.40)], we have

$$
\begin{aligned}
P(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}
&= \int d^2\lambda\, P(\alpha,\alpha^*,t|\lambda,\lambda^*,0)P(\lambda,\lambda^*,0)_{\rho(0)=|l\rangle\langle l|}\\
&= \int d^2\lambda\, \frac{1}{\pi\bar n(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha-\lambda e^{-\frac{\gamma}{2}t}e^{-i\omega_0 t}|^2}{\bar n(1-e^{-\gamma t})}\right]\\
&\quad\times \frac{1}{l!}e^{|\lambda|^2}\frac{\partial^{2l}}{\partial\lambda^l\partial\lambda^{*l}}\delta(\lambda)\\
&= \frac{1}{l!}\frac{\partial^{2l}}{\partial\lambda^l\partial\lambda^{*l}}\left\{\frac{1}{\pi\bar n(1-e^{-\gamma t})}\right.\\
&\quad\left.\left.\times\exp\left[-\frac{|\alpha-\lambda e^{-\frac{\gamma}{2}t}e^{-i\omega_0 t}|^2}{\bar n(1-e^{-\gamma t})}\right]e^{|\lambda|^2}\right\}\right|_{\lambda=\lambda^*=0},
\end{aligned}
$$

where the integration is performed using (3.37). Expanding the function inside the curly bracket,

$$
\begin{aligned}
P(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}
&= \frac{1}{\pi\bar n(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha|^2}{\bar n(1-e^{-\gamma t})}\right]\\
&\quad\times\frac{1}{l!}\frac{\partial^{2l}}{\partial\lambda^l\partial\lambda^{*l}}\left\{\exp\left[-|\lambda|^2\frac{e^{-\gamma t}-\bar n(1-e^{-\gamma t})}{\bar n(1-e^{-\gamma t})}\right]\right.\\
&\quad\left.\left.\times\exp\left[\lambda\frac{\alpha^* e^{-(\gamma/2)t}e^{-i\omega_0 t}}{\bar n(1-e^{-\gamma t})}\right]\exp\left[\lambda^*\frac{\alpha e^{-(\gamma/2)t}e^{i\omega_0 t}}{\bar n(1-e^{-\gamma t})}\right]\right\}\right|_{\lambda=\lambda^*=0}.
\end{aligned}
$$

The derivatives can be evaluated using (4.46), with

$$
A=\frac{e^{-\gamma t}-\bar n(1-e^{-\gamma t})}{\bar n(1-e^{-\gamma t})}\quad\text{and}\quad B^*=C=\frac{\alpha e^{-(\gamma/2)t}e^{i\omega_0 t}}{\bar n(1-e^{-\gamma t})}:
$$

the *P distribution for a damped Fock state* is then

$$
\begin{aligned}
P(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}
&= \frac{1}{\pi\bar n(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha|^2}{\bar n(1-e^{-\gamma t})}\right]\frac{1}{l!}\left[\frac{e^{-\gamma t}-\bar n(1-e^{-\gamma t})}{\bar n(1-e^{-\gamma t})}\right]^l\\
&\quad\times\sum_{k=0}^{l}(-1)^{l-k}\frac{l!}{k!(l-k)!}\frac{l!}{k!}\left\{\frac{|\alpha|^2e^{-\gamma t}}{\bar n(1-e^{-\gamma t})[e^{-\gamma t}-\bar n(1-e^{-\gamma t})]}\right\}^k.
\end{aligned}
\tag{4.50}
$$

In the long-time limit this expression clearly approaches the Gaussian describing a thermal state with mean photon number $\bar n$. This asymptotic solution is, of course, independent of the oscillator's initial state. To follow the evolution of $P(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}$ for short times, it is helpful to rewrite (4.50) in an alternative form. We define

$$
A=\frac{e^{-\gamma t}-\bar n(1-e^{-\gamma t})}{\bar n(1-e^{-\gamma t})}\quad\text{and}\quad\lambda=\frac{\alpha e^{-(\gamma/2)t}}{e^{-\gamma t}-\bar n(1-e^{-\gamma t})},
$$

and then (4.50) reads

$$
\begin{aligned}
P(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}
&= \frac{1}{\pi\bar n(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha|^2}{\bar n(1-e^{-\gamma t})}\right]\frac{1}{l!}\sum_{k=0}^{l}(-1)^{l-k}\frac{l!}{k!(l-k)!}\\
&\quad\times\frac{l!}{k!}A^{l-k}(-A\lambda^*)^k(-A\lambda)^k.
\end{aligned}
$$

Equation (4.46) may now be used a second time, with $B=C=0$, to obtain

$$
P(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}
= \frac{1}{\pi\bar n(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha|^2}{\bar n(1-e^{-\gamma t})}\right]\frac{1}{l!}e^{A|\lambda|^2}\frac{\partial^{2l}}{\partial\lambda^l\partial\lambda^{*l}}e^{-A|\lambda|^2}.
$$

After resubstituting the explicit expressions for A and λ, we have an *alternative form for the P distribution for a damped Fock state*:

$$
\begin{aligned}
P(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}
&= \frac{1}{l!}\exp\left[\frac{|\alpha|^2}{e^{-\gamma t}-\bar n(1-e^{-\gamma t})}\right]\left[\frac{e^{-\gamma t}-\bar n(1-e^{-\gamma t})}{e^{-(\gamma/2)t}}\right]^{2l}\\
&\quad\times\frac{\partial^{2l}}{\partial\alpha^l\partial\alpha^{*l}}\left\{\frac{1}{\pi\bar n(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha|^2e^{-\gamma t}}{\bar n(1-e^{-\gamma t})[e^{-\gamma t}-\bar n(1-e^{-\gamma t})]}\right]\right\}.
\end{aligned}
\tag{4.51}
$$

From this expression

$$
P(\alpha,\alpha^*,0)_{\rho(0)=|l\rangle\langle l|}=\frac{1}{l!}e^{|\alpha|^2}\frac{\partial^{2l}}{\partial\alpha^l\partial\alpha^{*l}}\left\{\lim_{t\to0+}\left(\frac{1}{\pi\bar n\gamma t}e^{-|\alpha|^2/\bar n\gamma t}\right)\right\}.\tag{4.52}
$$

Equation (4.52) shows explicitly the time-reversed approach ($t\to0+$) of $P(\alpha,\alpha^*,t)$ to its initial form in terms of derivatives of a two-dimensional δ-function.

Note that if $\bar n\neq0$, $P(\alpha,\alpha^*,t)$ is actually a well-behaved function for all times $t>0$. Thermal fluctuations destroy the singular character of the initial Fock state as soon as the interaction with the reservoir is turned on: for short times the singular distribution representing the initial Fock state is replaced by a derivative (of order $2l$) of a very narrow Gaussian whose variance is growing linearly with time. Nonetheless, $P(\alpha,\alpha^*,t)$ remains unacceptable as a classical probability distribution for a finite time after $t=0$. During the early part of its evolution it takes on negative values – for example, for $l=1$, (4.50) has the form

$$P(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|} = \frac{1}{\pi\bar{n}(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha|^2}{\bar{n}(1-e^{-\gamma t})}\right] \times\left\{1-\frac{e^{-\gamma t}}{\bar{n}(1-e^{-\gamma t})}+\frac{|\alpha|^2 e^{-\gamma t}}{[\bar{n}(1-e^{-\gamma t})]^2}\right\}, \quad (4.53)$$

This distribution takes negative values inside the circle $|\alpha|^2 = \bar{n}(1-e^{-\gamma t})[1-\bar{n}(e^{\gamma t}-1)]$ during the time interval $0<\gamma t<\ln(\bar{n}+1)-\ln\bar{n}$.

Exercise 4.4 Show that (4.50) gives

$$\langle(a^\dagger a)(t)\rangle = \left(\overline{\alpha^*\alpha(t)}\right)_P = le^{-\gamma t}+\bar{n}(1-e^{-\gamma t}),$$

in agreement with (1.70).

4.2.3 Damped Fock State in the Q and Wigner Representations

We have seen that the Q distribution is proportional to the diagonal matrix elements of ρ in the coherent state basis, and therefore it cannot become negative [Eq. (4.6)]. Indeed, the Green function (4.16) and the distribution (4.10) representing an initial Fock state in the Q representation are everywhere positive; it is clear then that $Q(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}$ for a damped Fock state will be nonnegative at all times. To calculate this distribution explicitly we use (4.16) and (4.10) to write

$$\begin{aligned}
&Q(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}\\
&= \int d^2\lambda\, Q(\alpha,\alpha^*,t|\lambda,\lambda^*,0)Q(\lambda,\lambda^*,0)_{\rho(0)=|l\rangle\langle l|}\\
&= \int d^2\lambda\,\frac{1}{\pi(\bar{n}+1)(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha-\lambda e^{-\frac{\gamma}{2}t}e^{-i\omega_0 t}|^2}{(\bar{n}+1)(1-e^{-\gamma t})}\right]\frac{1}{\pi}e^{-|\lambda|^2}\frac{|\lambda|^{2l}}{l!}\\
&= \frac{1}{\pi(\bar{n}+1)(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha|^2}{(\bar{n}+1)(1-e^{-\gamma t})}\right]\\
&\quad\times\frac{1}{\pi}\frac{1}{l!}\int d^2\lambda\,|\lambda|^{2l}\exp\left[-|\lambda|^2\frac{e^{-\gamma t}+(\bar{n}+1)(1-e^{-\gamma t})}{(\bar{n}+1)(1-e^{-\gamma t})}\right]\\
&\quad\times\exp\left[\frac{\alpha\lambda^* e^{-(\gamma/2)t}e^{-i\omega_0 t}+\alpha^*\lambda e^{-(\gamma/2)t}e^{i\omega_0 t}}{(\bar{n}+1)(1-e^{-\gamma t})}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi(\bar{n}+1)(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha|^2}{(\bar{n}+1)(1-e^{-\gamma t})}\right]\\
&\quad\times\frac{1}{\pi}\frac{1}{l!}\int_0^\infty dr\,r^{2l+1}\exp\left[-r^2\frac{e^{-\gamma t}+(\bar{n}+1)(1-e^{-\gamma t})}{(\bar{n}+1)(1-e^{-\gamma t})}\right]\\
&\quad\times\int_0^{2\pi}d\phi\,\exp\left[\frac{2|\alpha|e^{-(\gamma/2)t}}{(\bar{n}+1)(1-e^{-\gamma t})}r\cos\phi\right],
\end{aligned}$$

where $r\equiv|\lambda|$, and $\phi\equiv\arg(\lambda)-\arg(\alpha)+\omega_0 t$. The angular integral gives a Bessel function. With this Bessel function expressed in its series representation we find

$$\begin{aligned}
&Q(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}\\
&= \frac{1}{\pi(\bar{n}+1)(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha|^2}{(\bar{n}+1)(1-e^{-\gamma t})}\right]\\
&\quad\times\frac{1}{\pi}\frac{1}{l!}\int_0^\infty dr\,r^{2l+1}\exp\left[-r^2\frac{e^{-\gamma t}+(\bar{n})+1)(1-e^{-\gamma t})}{(\bar{n}+1)(1-e^{-\gamma t})}\right]\\
&\quad\times 2\pi\sum_{k=0}^\infty\frac{1}{(k!)^2}\left[\frac{r|\alpha|e^{-(\gamma/2)t}}{(\bar{n}+1)(1-e^{-\gamma t})}\right]^{2k}\\
&= \frac{1}{\pi(\bar{n}+1)(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha|^2}{(\bar{n}+1)(1-e^{-\gamma t})}\right]\\
&\quad\times\frac{1}{l!}\sum_{k=0}^\infty\frac{1}{(k!)^2}\left[\frac{|\alpha|e^{-(\gamma/2)t}}{(\bar{n}+1)(1-e^{-\gamma t})}\right]^{2k}\\
&\quad\times 2\int_0^\infty dr\,r^{2(k+l)+1}\exp\left[-r^2\frac{1+\bar{n}(1-e^{-\gamma t})}{(\bar{n}+1)(1-e^{-\gamma t})}\right].
\end{aligned}$$

The remaining integral is performed by repeated integration by parts and gives

$$\begin{aligned}
&Q(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}\\
&= \frac{1}{\pi(\bar{n}+1)(1-e^{-\gamma t})}\exp\left[-\frac{|\alpha|^2}{(\bar{n}+1)(1-e^{-\gamma t})}\right]\\
&\quad\times\frac{1}{l!}\sum_{k=0}^\infty\frac{1}{(k!)^2}\left[\frac{|\alpha|e^{-(\gamma/2)t}}{(\bar{n}+1)(1-e^{-\gamma t})}\right]^{2k}(k+l)!\left[\frac{(\bar{n}+1)(1-e^{-\gamma t})}{1+\bar{n}(1-e^{-\gamma t})}\right]^{k+l+1}.
\end{aligned}$$

The Q *distribution for a damped Fock state* is then

$$Q(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}$$
$$= \frac{1}{\pi[1+\bar{n}(1-e^{-\gamma t})]}\exp\left[-\frac{|\alpha|^2}{(\bar{n}+1)(1-e^{-\gamma t})}\right]\frac{1}{l!}\left[\frac{(\bar{n}+1)(1-e^{-\gamma t})}{1+\bar{n}(1-e^{-\gamma t})}\right]^l$$
$$\times\sum_{k=0}^{\infty}\frac{(k+l)!}{(k!)^2}\left\{\frac{|\alpha|^2e^{-\gamma t}}{(\bar{n}+1)(1-e^{-\gamma t})[1+\bar{n}(1-e^{-\gamma t})]}\right\}^k. \tag{4.54}$$

Again, this expression clearly shows the evolution to a Gaussian distribution describing a thermal state in the long-time limit – now with the increased variance ($\bar{n}\to\bar{n}+1$) discussed below (4.21). Our result does not have the most convenient form, however, since the summation includes an infinite number of divergent terms in the limit $t\to 0$. Of course, $Q(\alpha,\alpha^*,0)$ does not diverge; this is prevented by the exponential multiplying the sum. It would be nice to have a form that cancels the divergent sum explicitly to reproduce the Q distribution for the initial Fock state in an obvious way. This can be accomplished using the following result:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(k+l)!}{(k!)^2}x^k &= \frac{d^l}{dx^l}\left(\sum_{k=0}^{\infty}\frac{1}{k!}x^{k+l}\right)\\ &= \frac{d^l}{dx^l}(x^le^x)\\ &= \sum_{k=0}^{l}\frac{l!}{k!(l-k)!}\frac{l!}{(l-k)!}x^{l-k}\frac{d^{l-k}}{dx^{l-k}}e^x\\ &= e^x\sum_{k=0}^{l}\frac{l!}{k!(l-k)!}\frac{l!}{k!}x^k.\end{aligned} \tag{4.55}$$

The third line follows from (4.45), with $A=-1$, $B=C=0$, and $n=1$; also, in the last line we have changed the summation index, with $l-k\to k$. Using (4.55), equation (4.54) may be recast to give an *alternative form for the Q distribution for a damped Fock state*:

$$Q(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}$$
$$= \frac{1}{\pi[1+\bar{n}(1-e^{-\gamma t})]}\exp\left[-\frac{|\alpha|^2}{1+\bar{n}(1-e^{-\gamma t})}\right]\frac{1}{l!}\left[\frac{(\bar{n}+1)(1-e^{-\gamma t})}{1+\bar{n}(1-e^{-\gamma t})}\right]^l$$
$$\times\sum_{k=0}^{l}\frac{l!}{k!(l-k)!}\frac{l!}{k!}\left\{\frac{|\alpha|^2e^{-(\gamma/2)t}}{(\bar{n}+1)(1-e^{-\gamma t})[1+\bar{n}(1-e^{-\gamma t})]}\right\}^k. \tag{4.56}$$

Equation (4.56) produces the correct initial distribution in an obvious way (only the $k=l$ term in the sum survives), and it also produces the Gaussian form in the long-time limit. It is clearly everywhere positive; for example, for $l=1$,

$$Q(\alpha,\alpha^*,t)_{\rho(0)=|1\rangle\langle 1|} = \frac{1}{\pi[1+\bar{n}(1-e^{-\gamma t})]}\exp\left[-\frac{|\alpha|^2}{1+\bar{n}(1-e^{-\gamma t})}\right]$$
$$\times\left\{1+\frac{|\alpha|^2e^{-\gamma t}}{[1+\bar{n}(1-e^{-\gamma t})]^2}\right\}, \tag{4.57}$$

which is to be compared with the result (4.53) for the corresponding P distribution.

Exercise 4.5 The Wigner distribution can be derived in a similar manner. Show that the *Wigner distribution for a damped Fock state* is given by

$$W(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}$$
$$= \frac{2}{\pi[1+2\bar{n}(1-e^{-\gamma t})]}\exp\left[-\frac{2|\alpha|^2}{1+2\bar{n}(1-e^{-\gamma t})}\right]$$
$$\times\frac{1}{l!}\sum_{k=0}^{l}(-1)^{l-k}\frac{l!}{k!(l-k)!}\frac{l!}{k!}2^{2k}\left[\frac{(\bar{n}+\frac{1}{2})(1-e^{-\gamma t})}{1+2\bar{n}(1-e^{-\gamma t})}\right]^k$$
$$\times\sum_{r=0}^{k}\frac{k!}{r!(k-r)!}\frac{k!}{r!}\left\{\frac{|\alpha|^2e^{-\gamma t}}{(\bar{n}+\frac{1}{2})(1-e^{-\gamma t})[1+2\bar{n}(1-e^{-\gamma t})]}\right\}^r. \tag{4.58}$$

Like $P(\alpha,\alpha^*,t)_{\rho(0)=|l\rangle\langle l|}$, this distribution can be negative. Analyze its behavior for $l=1$.

4.3 Two-Time Averages

In Sect. 1.5 we obtained expressions for calculating two-time averages from an operator master equation. We have now seen that the operator master equation can be converted into a partial differential equation – in the case of the damped harmonic oscillator, a Fokker–Planck equation – by setting up a correspondence between ρ and a phase-space distribution function. How can the formal operator expressions given in Sect. 1.5 be cast into phase-space language to allow us to calculate two-time averages at the "classical" end of the quantum–classical correspondence? This is the question we now address. Answering the question in a general way requires that we first develop a little more formalism. The notation of this formalism is itself a bit burdensome, and certainly some of the calculations we eventually perform with it are rather arcane. It is perhaps helpful, then, to look ahead to (4.100a) and (4.100b). These state the result used most widely in applications; namely, that normal-ordered, time-ordered two-time averages, such as those needed to calculation an optical spectrum or intensity correlation function, are given by phase-space integrals in the P representation analogous to those met in classical statistics. The effort expended with the formalism allows us to generalize from this result in two directions: to determine which two-time averages are given by similar phase-space integrals in the Q and Wigner representations, and to see how derivatives of the phase-space distribution must be taken, as in Sec. 4.1.3, if inappropriately ordered operator averages are considered.

4.3.1 Quantum–Classical Correspondence for General Operators

Consider the relationship defined by (3.70) and (3.72) between the operator ρ and the distribution $P(\alpha,\alpha^*)$. There is actually no reason to restrict this relationship to density operators; we can generalize it to set up a correspondence between any system operator $\hat{O}$ and a function $F_{\hat{O}}^{(a)}(\alpha,\alpha^*)$ (we use "function" remembering that this may be a generalized function). As a generalization of the characteristic function $\chi_N(z,z^*)$ we define

$$\tilde{F}_{\hat{O}}^{(a)}(z,z^*) \equiv \pi\mathrm{tr}\big(\hat{O}e^{iz^*a^\dagger}e^{iza}\big); \tag{4.59}$$

the generalization of the P distribution is then

$$F_{\hat{O}}^{(a)}(\alpha,\alpha^*) \equiv \frac{1}{\pi^2}\int d^2z\,\tilde{F}_{\hat{O}}^{(a)}(z,z^*)e^{-iz^*\alpha^*}e^{-iz\alpha}, \tag{4.60}$$

with the inverse relationship

$$\tilde{F}_{\hat{O}}^{(a)}(z,z^*) = \int d^2\alpha\,F_{\hat{O}}^{(a)}(\alpha,\alpha^*)e^{iz^*\alpha^*}e^{iz\alpha}. \tag{4.61}$$

Taken together (4.59) and (4.60) set up a correspondence between the operator $\hat{O}$ and the phase-space function $F_{\hat{O}}^{(a)}(\alpha,\alpha^*)$. In place of the relationship that gives normal-ordered moments in the P representation [Eqs. (3.71) and (3.74)] we now have the more general result

$$\begin{aligned}\mathrm{tr}\big(\hat{O}a^{\dagger p}a^q\big) &= \frac{1}{\pi}\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\tilde{F}_{\hat{O}}^{(a)}(z,z^*)\bigg|_{z=z^*=0}\\ &= \frac{1}{\pi}\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\int d^2\alpha\,F_{\hat{O}}^{(a)}(\alpha,\alpha^*)e^{iz^*\alpha^*}e^{iz\alpha}\bigg|_{z=z^*=0}\\ &= \frac{1}{\pi}\int d^2\alpha\,F_{\hat{O}}^{(a)}(\alpha,\alpha^*)\alpha^{*p}\alpha^q.\end{aligned} \tag{4.62}$$

Within this scheme the P distribution is defined with

$$\chi_N(z,z^*) \equiv \frac{1}{\pi}\tilde{F}_\rho^{(a)}(z,z^*), \tag{4.63a}$$

$$P(\alpha,\alpha^*) \equiv \frac{1}{\pi}F_\rho^{(a)}(\alpha,\alpha^*). \tag{4.63b}$$

We have slipped in some changes here that need an explanation: a factor of π has been added in (4.59) and the subscript N on χ_N has been replaced by the superscript (a) on $\tilde{F}_\rho^{(a)}$. This has been done with the following in mind.

Consider an operator $\hat{A}$ expanded as a power series of terms written in antinormal order:

$$\hat{A} = A(a,a^\dagger) \equiv \sum_{p,q} C_{p,q}^{(a)}a^q a^{\dagger p}, \tag{4.64}$$

where the $C_{p,q}^{(a)}$ are constants. Then, from (4.59),

$$\begin{aligned}\tilde{F}_{\hat{A}}^{(a)}(z,z^*) &= \pi\sum_{p,q}C_{p,q}^{(a)}\mathrm{tr}\big(a^{\dagger p}e^{iz^*a^\dagger}e^{iza}a^q\big)\\ &= \pi\sum_{p,q}C_{p,q}^{(a)}\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\mathrm{tr}\big(e^{iz^*a^\dagger}e^{iza}\big).\end{aligned}$$

Introducing the expansion (3.9) for the unit operator,

$$\begin{aligned}\tilde{F}_{\hat{A}}^{(a)}(z,z^*) &= \pi\sum_{p,q}C_{p,q}^{(a)}\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\mathrm{tr}\left(\frac{1}{\pi}\int d^2\lambda\,|\lambda\rangle\langle\lambda|e^{iz^*a^\dagger}e^{iza}\right)\\ &= \sum_{p,q}C_{p,q}^{(a)}\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\int d^2\lambda\,e^{iz^*\lambda^*}e^{iz\lambda}\\ &= \pi^2\sum_{p,q}C_{p,q}^{(a)}\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\delta(z).\end{aligned}$$

We substitute this result into (4.60) and integrate by parts to obtain

$$\begin{aligned}F_{\hat{A}}^{(a)}(\alpha,\alpha^*) &= \sum_{p,q}C_{p,q}^{(a)}\int d^2z\left[\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\delta(z)\right]e^{-iz^*\alpha^*}e^{-iz\alpha}\\ &= \sum_{p,q}C_{p,q}^{(a)}\int d^2z\,\delta(z)\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}e^{-iz^*\alpha^*}e^{-iz\alpha}.\end{aligned}$$

Thus,

$$F_{\hat{A}}^{(a)}(\alpha,\alpha^*) = \sum_{p,q}C_{p,q}^{(a)}\alpha^{*p}\alpha^q = A(\alpha,\alpha^*). \tag{4.65}$$

Equations (4.64) and (4.65) state that, for operators written as an antinormal-ordered series, $F_{\hat{O}}^{(a)}(\alpha,\alpha^*)$ is obtained by replacing the operators a and $a^\dagger$ in that series by the complex numbers α and α^*, respectively. $F_{\hat{O}}^{(a)}(\alpha,\alpha^*)$ is called the *antinormal-ordered associated function* for the operator $\hat{O}$. The superscript (a) denotes the *antinormal-ordered* associated function. The factor of π in (4.59) leads to the direct association of functions and operators expressed by (4.64) and (4.65), rather than with a $1/\pi$ multiplying the right-hand side of (4.65). We must be careful now not to become confused between our "normals" and "antinormals". In (4.63b) we see that $P(\alpha,\alpha^*)$, which is used to calculate *normal-ordered* averages, is, apart from a factor of π, the *antinormal-ordered* associated function for ρ. This relationship will become clearer as we follow the idea of associated functions a little further.

Analogous definitions of *normal-ordered and symmetrically ordered associated functions* for an operator can be given. We define the normal-ordered associated function $F_{\hat{O}}^{(n)}(\alpha,\alpha^*)$ in terms of its Fourier transform $\tilde{F}_{\hat{O}}^{(n)}(z,z^*)$ introduced as a generalization of (4.1): We define

$$\tilde{F}_{\hat{O}}^{(n)}(z,z^*) \equiv \pi\mathrm{tr}\big(\hat{O}e^{iza}e^{iz^*a^\dagger}\big), \tag{4.66}$$

and

$$F_{\hat{O}}^{(n)}(\alpha,\alpha^*) \equiv \frac{1}{\pi^2}\int d^2z\,\tilde{F}_{\hat{O}}^{(n)}(z,z^*)e^{-iz^*\alpha^*}e^{-iz\alpha}, \tag{4.67}$$

with the inverse relationship

$$\tilde{F}_{\hat{O}}^{(n)}(z,z^*) = \int d^2\alpha\,F_{\hat{O}}^{(n)}(\alpha,\alpha^*)e^{iz^*\alpha^*}e^{iz\alpha}. \tag{4.68}$$

In place of the relationship that gives antinormal-ordered moments in the Q representation [Eq. (4.5)], we have

$$\mathrm{tr}\big(\hat{O}a^q a^{\dagger p}\big) = \frac{1}{\pi}\int d^2\alpha\,F_{\hat{O}}^{(n)}(\alpha,\alpha^*)\alpha^{*p}\alpha^q. \tag{4.69}$$

The Q distribution is proportional to the normal-ordered associated function for ρ:

$$\chi_A(z,z^*) \equiv \frac{1}{\pi}\tilde{F}_\rho^{(n)}(z,z^*), \tag{4.70a}$$

$$Q(\alpha,\alpha^*) \equiv \frac{1}{\pi}F_\rho^{(n)}(\alpha,\alpha^*). \tag{4.70b}$$

Similarly, the symmetric-ordered associated function $F_{\hat{O}}^{(s)}(\alpha,\alpha^*)$ is defined in terms of its Fourier transform $\tilde{F}_{\hat{O}}^{(s)}(z,z^*)$ introduced as a generalization of (4.25): We define

$$\tilde{F}_{\hat{O}}^{(s)}(z,z^*) \equiv \pi\mathrm{tr}\big(\hat{O}e^{iz^*a^\dagger+iza}\big), \tag{4.71}$$

and

$$F_{\hat{O}}^{(s)}(\alpha,\alpha^*) \equiv \frac{1}{\pi^2}\int d^2z\,\tilde{F}_{\hat{O}}^{(s)}(z,z^*)e^{-iz^*\alpha^*}e^{-iz\alpha}. \tag{4.72}$$

with the inverse relationship

$$\tilde{F}_{\hat{O}}^{(s)}(z,z^*) = \int d^2\alpha\,F_{\hat{O}}^{(s)}(\alpha,\alpha^*)e^{iz^*\alpha^*}e^{iz\alpha}. \tag{4.73}$$

In place of the relationship that gives symmetric-ordered moments in the Wigner representation [Eq. (4.31)], we have

$$\mathrm{tr}\Big[\hat{O}\big(a^{\dagger p}a^q\big)_S\Big] = \frac{1}{\pi}\int d^2\alpha\,F_{\hat{O}}^{(s)}(\alpha,\alpha^*)\alpha^{*p}\alpha^q. \tag{4.74}$$

The Wigner distribution is proportional to the symmetric-ordered associated function for ρ:

$$\chi_S(z,z^*) \equiv \frac{1}{\pi}\tilde{F}_\rho^{(s)}(z,z^*), \tag{4.75a}$$

$$W(\alpha,\alpha^*) \equiv \frac{1}{\pi}F_\rho^{(s)}(\alpha,\alpha^*). \tag{4.75b}$$

Relationships between the various associated functions, and between their Fourier transforms, can be obtained as generalizations of earlier results: equations (4.9) and (4.34) generalize to give

$$\tilde{F}_{\hat{O}}^{(n)}(z,z^*) = e^{-\frac{1}{2}|z|^2}\tilde{F}_{\hat{O}}^{(s)}(z,z^*) = e^{-|z|^2}\tilde{F}_{\hat{O}}^{(a)}(z,z^*); \tag{4.76}$$

Eqs. (4.7) and (4.35) generalize to give

$$F_{\hat{O}}^{(n)}(\alpha,\alpha^*) = \frac{1}{\pi}\int d^2\lambda\,e^{-|\lambda-\alpha|^2}F_{\hat{O}}^{(a)}(\lambda,\lambda^*), \tag{4.77a}$$

$$F_{\hat{O}}^{(s)}(\alpha,\alpha^*) = \frac{2}{\pi}\int d^2\lambda\,e^{-2|\lambda-\alpha|^2}F_{\hat{O}}^{(a)}(\lambda,\lambda^*), \tag{4.77b}$$

$$F_{\hat{O}}^{(n)}(\alpha,\alpha^*) = \frac{2}{\pi}\int d^2\lambda\,e^{-2|\lambda-\alpha|^2}F_{\hat{O}}^{(s)}(\lambda,\lambda^*); \tag{4.77c}$$

finally, Eqs. (4.11) and (4.36) generalize to give

$$F_{\hat{O}}^{(n)}(\alpha,\alpha^*) = \exp\left(\frac{1}{2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)F_{\hat{O}}^{(s)}(\alpha,\alpha^*) = \exp\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)F_{\hat{O}}^{(a)}(\alpha,\alpha^*). \tag{4.78}$$

We can now understand the relationships between the various associated functions for ρ (the P, Q and Wigner distributions) and the ordered operator averages that are calculated from their moments in a more general context. First, we note the extension of the result expressed by (4.64) and (4.65) to normal-ordered and symmetric-ordered series. For an operator $\hat{N}$ written as a normal-ordered series,

$$\hat{N} = N(a,a^\dagger) \equiv \sum_{p,q}C_{p,q}^{(n)}a^{\dagger p}a^q, \tag{4.79}$$

the normal-ordered associated function is obtained by replacing a by α and $a^\dagger$ by α^*:

$$F_{\hat{N}}^{(n)}(\alpha,\alpha^*) = \sum_{p,q}C_{p,q}^{(n)}\alpha^{*p}\alpha^q = N(\alpha,\alpha^*). \tag{4.80}$$

For an operator $\hat{S}$ written as a symmetric-ordered series,

$$\hat{S} = S(a, a^\dagger) \equiv \sum_{p,q} C^{(s)}_{p,q} (a^{\dagger p} a^q)_S, \tag{4.81}$$

the symmetric-ordered associated function is obtained by replacing a by α and $a^\dagger$ by α^*:

$$F^{(s)}_{\hat{S}}(\alpha, \alpha^*) = \sum_{p,q} C^{(s)}_{p,q} \alpha^{*p} \alpha^q = S(\alpha, \alpha^*). \tag{4.82}$$

Now, if $\hat{O}_1$ and $\hat{O}_2$ are arbitrary system operators, and $\hat{N}_2 = N_2(a, a^\dagger) = \hat{O}_2$ is the normal-ordered form of $\hat{O}_2$, we can apply (4.62) to each term in the series expansion of $N_2(a, a^\dagger)$ to obtain

$$\begin{aligned}\mathrm{tr}(\hat{O}_1 \hat{O}_2) &= \mathrm{tr}[\hat{O}_1 N_2(a, a^\dagger)] \\ &= \frac{1}{\pi} \int d^2\alpha\, F^{(a)}_{\hat{O}_1}(\alpha, \alpha^*) N_2(\alpha, \alpha^*) \\ &= \frac{1}{\pi} \int d^2\alpha\, F^{(a)}_{\hat{O}_1}(\alpha, \alpha^*) F^{(n)}_{\hat{O}_2}(\alpha, \alpha^*),\end{aligned} \tag{4.83}$$

where the last line follows from (4.80). Equations (3.74) and (4.5), giving normal-ordered and antinormal-ordered operator averages as moments of the P and Q distributions, respectively, are special cases of this more general result. With $\hat{O}_1$ taken as ρ, moments of the *antinormal-ordered* associated function for ρ give the averages of operators $\hat{O}_2$ written in *normal-ordered* form. Alternatively, with $\hat{O}_2$ taken as ρ, moments of the *normal-ordered* associated function for ρ give averages of operators $\hat{O}_1$ written in *antinormal-ordered* form. A similar result can be obtained by writing $\hat{O}_2$ as a symmetric-ordered series and using (4.74) and (4.82):

$$\mathrm{tr}(\hat{O}_1 \hat{O}_2) = \frac{1}{\pi} \int d^2\alpha\, F^{(s)}_{\hat{O}_1}(\alpha, \alpha^*) F^{(s)}_{\hat{O}_2}(\alpha, \alpha^*). \tag{4.84}$$

The relationship (4.31) between symmetric-ordered operator averages and the moments of the Wigner distribution is a special case of this result.

Note 4.5 The association given by (4.79) and (4.80) is easily proved following an argument analogous to that used to establish (4.65). A similar proof of the association given by (4.81) and (4.82) is not so straightforward because partial derivatives with respect to (iz) and (iz^*) act in a rather complicated way on $e^{iz^* a^\dagger + iza}$ (see Sect. 4.3.5). A simple proof can be devised, however, by arguing backwards as follows: Set $F^{(s)}_{\hat{O}}(\alpha, \alpha^*) = \alpha^{*p} \alpha^q$. What, then, is the operator $\hat{O}$ having this symmetric-ordered associated function? The answer to this question can be obtained by converting everything into normal order, using (4.78) to write

$$\begin{aligned}F^{(n)}_{\hat{O}}(\alpha, \alpha^*) &= \exp\left(\frac{1}{2} \frac{\partial^2}{\partial\alpha \partial\alpha^*}\right) \alpha^{*p} \alpha^q \\ &= \sum_{k=0}^{\min(p,q)} \frac{1}{2^k} \frac{1}{k!} \frac{p!}{(p-k)!} \frac{q!}{(q-k)!} \alpha^{*p-k} \alpha^{q-k}.\end{aligned}$$

Then, from (4.79) and (4.80),

$$\hat{O} = \sum_{k=0}^{\min(p,q)} \frac{1}{2^k} \frac{1}{k!} \frac{p!}{(p-k)!} \frac{q!}{(q-k)!} a^{\dagger p-k} a^{q-k}.$$

But (4.33) tells us that this is just the symmetric-ordered operator $(a^{\dagger p} a^q)_S$.

4.3.2 Associated Functions and the Master Equation

We saw how to derive an equation of motion for the P distribution to replace the operator master equation in Sect. 3.2.2. Generally, we will refer to such an equation as a *phase-space equation of motion*. We now see what this equation of motion looks like in the language of our generalized formalism of associated functions for arbitrary operators.

Let us start with a rather formal summary of the derivation of the equation of motion for the P distribution. From the operator master equation (3.1) we write

$$\frac{\partial}{\partial t} \mathrm{tr}\left[\rho(t) e^{iz^* a^\dagger} e^{iza}\right] = \mathrm{tr}\left[\left(\mathcal{L}\rho(t)\right) e^{iz^* a^\dagger} e^{iza}\right], \tag{4.85}$$

which, after substituting the explicit form of $\mathcal{L}$ for the damped harmonic oscillator, is just (3.76). In the language of associated functions (4.85) states that

$$\frac{\partial}{\partial t} \tilde{F}^{(a)}_{\rho(t)}(z, z^*) = \tilde{F}^{(a)}_{\mathcal{L}\rho(t)}(z, z^*). \tag{4.86}$$

The Fourier transform of this equation gives the equation of motion for the antinormal-ordered associated function for ρ – the P distribution (multiplied by π):

$$\frac{\partial}{\partial t} F^{(a)}_{\rho(t)}(\alpha, \alpha^*) = F^{(a)}_{\mathcal{L}\rho(t)}(\alpha, \alpha^*). \tag{4.87}$$

Formally, this is the Fokker–Planck equation. But the next step is needed to reveal its explicit form as a partial differential equation; this is the step where most of our effort was spent in Sect. 3.2.2. We must express $F^{(a)}_{\mathcal{L}\rho(t)}(\alpha, \alpha^*)$ in terms of $F^{(a)}_{\rho(t)}(\alpha, \alpha^*)$, with the action of $\mathcal{L}$ on the density operator ρ transformed into the action of some differential operator on the associated function for ρ. Leaving out the details, the aim is to write

$$F^{(a)}_{\mathcal{L}\rho(t)}(\alpha, \alpha^*) = L^{(a)}\left(\alpha, \alpha^*, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*}\right) F^{(a)}_{\rho(t)}(\alpha, \alpha^*), \tag{4.88}$$

where $L^{(a)}(\alpha, \alpha^*, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*})$ is a differential operator associated with $\mathcal{L}$. For any particular example this must be found from an explicit calculation similar to the one in Sect. 3.2.2; for the damped harmonic oscillator

$$L^{(a)}\Big(\alpha, \alpha^*, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*}\Big) = \Big(\frac{\gamma}{2} + i\omega_0\Big)\frac{\partial}{\partial\alpha}\alpha + \Big(\frac{\gamma}{2} - i\omega_0\Big)\frac{\partial}{\partial\alpha^*}\alpha^* + \gamma\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}. \tag{4.89}$$

Now (4.87) becomes

$$\frac{\partial}{\partial t}F^{(a)}_{\rho(t)}(\alpha, \alpha^*) = L^{(a)}\Big(\alpha, \alpha^*, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*}\Big)F^{(a)}_{\rho(t)}(\alpha, \alpha^*), \tag{4.90}$$

and setting

$$P(\alpha, \alpha^*, t) = \frac{1}{\pi}F^{(a)}_{\rho(t)}(\alpha, \alpha^*), \tag{4.91}$$

the equation of motion for $P(\alpha, \alpha^*, t)$ is

$$\frac{\partial}{\partial t}P(\alpha, \alpha^*, t) = L^{(a)}\Big(\alpha, \alpha^*, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*}\Big)P(\alpha, \alpha^*, t). \tag{4.92}$$

More generally, we may write (4.88), not just for density operators, but for any operator $\hat{O}$. Then, by induction,

$$F^{(a)}_{\mathcal{L}^k\hat{O}}(\alpha, \alpha^*) = \Big[L^{(a)}\Big(\alpha, \alpha^*, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*}\Big)\Big]^k F^{(a)}_{\hat{O}}(\alpha, \alpha^*), \tag{4.93}$$

from which it follows that

$$F^{(a)}_{\exp(\mathcal{L}\tau)\hat{O}}(\alpha, \alpha^*) = e^{L^{(a)}(\alpha, \alpha^*, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*})\tau}F^{(a)}_{\hat{O}}(\alpha, \alpha^*). \tag{4.94}$$

This result, and (4.83) from the last section, will serve as centerpieces in our conversion of the expressions from Sect. 1.5 for two-time averages into phase-space form.

Of course, we define the differential operators $L^{(n)}(\alpha, \alpha^*, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*})$ and $L^{(s)}(\alpha, \alpha^*, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*})$ which govern the dynamics of the Q and the Wigner distributions, respectively, in an analogous manner. For the damped harmonic oscillator $L^{(n)}(\alpha, \alpha^*, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*})$ is given by (4.89) with the replacement $\bar{n} \to \bar{n}+1$, and $L^{(s)}(\alpha, \alpha^*, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*})$ is given by the same expression with the replacement $\bar{n} \to \bar{n} + \frac{1}{2}$.

4.3.3 Normal-Ordered Time-Ordered Averages in the P Representation

We first set ourselves the task of finding a phase-space form in the P representation for the average ($\tau \geq 0$)

$$\langle a^{\dagger p}(t)\hat{N}(t+\tau)a^q(t)\rangle = \mathrm{tr}\big\{\big(e^{\mathcal{L}\tau}[a^q\rho(t)a^{\dagger p}]\big)\hat{N}\big\}, \tag{4.95}$$

where the expression on the right-hand side is obtained from (1.102); $\hat{N}$ can be any system operator written as a normal-ordered series [Eq. (4.79)]. Equation (4.95) provides an expression for calculating a general normal-ordered, time-ordered, two-time average – every $a^\dagger$ to the left of every a, every $a^\dagger(t+\tau)$ to the right of every $a^\dagger(t)$, and every $a(t+\tau)$ to the left of every $a(t)$. These are the averages that most interest us for applications in quantum optics.

Using (4.83) and (4.94), we write the average (4.95) as the phase-space integral

$$\begin{aligned}\langle a^{\dagger p}(t)\hat{N}(t+\tau)a^q(t)\rangle &= \frac{1}{\pi}\int d^2\alpha\, F^{(a)}_{\exp(\mathcal{L}\tau)[a^q\rho(t)a^{\dagger p}]}(\alpha, \alpha^*)\, F^{(n)}_{\hat{N}}(\alpha, \alpha^*)\\ &= \frac{1}{\pi}\int d^2\alpha \Big[e^{L^{(a)}(\alpha, \alpha^*, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\alpha^*})\tau}F^{(a)}_{a^q\rho(t)a^{\dagger p}}(\alpha, \alpha^*)\Big]F^{(n)}_{\hat{N}}(\alpha, \alpha^*).\end{aligned} \tag{4.96}$$

Then, from (4.60) and (4.59),

$$\begin{aligned}F^{(a)}_{a^q\rho(t)a^{\dagger p}}(\alpha, \alpha^*) &= \frac{1}{\pi^2}\int d^2z\, \tilde{F}^{(a)}_{a^q\rho(t)a^{\dagger p}}(z, z^*)e^{-iz^*\alpha^*}e^{-iz\alpha}\\ &= \frac{1}{\pi^2}\int d^2z\, \pi\mathrm{tr}\Big[a^q\rho(t)a^{\dagger p}e^{iz^*a^\dagger}e^{iza}\Big]e^{-iz^*\alpha^*}e^{-iz\alpha}\\ &= \frac{1}{\pi^2}\int d^2z\, \pi\mathrm{tr}\Big[\rho(t)a^{\dagger p}e^{iz^*a^\dagger}e^{iza}a^q\Big]e^{-iz^*\alpha^*}e^{-iz\alpha}\\ &= \frac{1}{\pi^2}\int d^2z\Big[\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\tilde{F}^{(a)}_{\rho(t)}(z, z^*)\Big]e^{-iz^*\alpha^*}e^{-iz\alpha}.\end{aligned}$$

Substituting for $\tilde{F}^{(a)}_{\rho(t)}(z, z^*)$ from (4.61), we have

$$\begin{aligned}F^{(a)}_{a^q\rho(t)a^{\dagger p}}(\alpha, \alpha^*) &= \frac{1}{\pi^2}\int d^2z\Big[\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\int d^2\lambda\, F^{(a)}_{\rho(t)}(\lambda, \lambda^*)e^{iz^*\lambda^*}e^{iz\lambda}\Big]\\ &\quad\times e^{-iz^*\alpha^*}e^{-iz\alpha}\\ &= \frac{1}{\pi^2}\int d^2\lambda\, F^{(a)}_{\rho(t)}(\lambda, \lambda^*)\lambda^{*p}\lambda^q\int d^2z\, e^{iz^*(\lambda^*-\alpha^*)}e^{iz(\lambda-\alpha)}\\ &= \frac{1}{\pi^2}\int d^2\lambda\, F^{(a)}_{\rho(t)}(\lambda, \lambda^*)\lambda^{*p}\lambda^q\delta^{(2)}(\lambda-\alpha)\\ &= F^{(a)}_{\rho(t)}(\alpha, \alpha^*)\alpha^{*p}\alpha^q.\end{aligned} \tag{4.97}$$

We now substitute this result into (4.96) to find ($\tau \geq 0$)

$$\langle a^{\dagger p}(t)\hat{N}(t+\tau)a^q(t)\rangle = \frac{1}{\pi}\int d^2\alpha \left[e^{L^{(a)}(\alpha,\alpha^*,\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\alpha^*})\tau} F^{(a)}_{\rho(t)}(\alpha,\alpha^*)\alpha^{*p}\alpha^q \right] F^{(n)}_{\hat{N}}(\alpha,\alpha^*). \tag{4.98}$$

At first sight, this expression may seem to be a rather useless formal result. However, a little more work casts it into a simple form – a form which might already have been anticipated. In simpler notation, (4.98) reads ($\tau \geq 0$)

$$\langle a^{\dagger p}(t)\hat{N}(t+\tau)a^q(t)\rangle = \int d^2\alpha \left[e^{L^{(a)}(\alpha,\alpha^*,\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\alpha^*})\tau} P(\alpha,\alpha^*,t)\alpha^{*p}\alpha^q \right] N(\alpha^*,\alpha), \tag{4.99}$$

where we have used (4.91) and (4.80). Now the action of the propagator $\exp\left[L^{(a)}(\alpha,\alpha^*,\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\alpha^*})\tau\right]$ on the δ-function $\delta^{(2)}(\alpha-\alpha_0)$ generates the Green function for the equation of motion (4.92). This suggests that we should write the operand of the propagator in (4.99) as

$$P(\alpha,\alpha^*,t)\alpha^{*p}\alpha^q = \int d^2\alpha_0\, \delta^{(2)}(\alpha-\alpha_0)P(\alpha_0,\alpha_0^*)\alpha_0^{*p}\alpha_0^q,$$

whence ($\tau \geq 0$), in the P representation a normal-ordered, time-ordered, two-time average is calculated as

$$\begin{aligned}\langle a^{\dagger p}(t)\hat{N}(t+\tau)a^q(t)\rangle &= \int d^2\alpha \int d^2\alpha_0\, \alpha_0^{*p}\alpha_0^q N(\alpha,\alpha^*)P(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0)P(\alpha_0,\alpha_0^*,t)\\ &= \left(\overline{(\alpha^{*p}\alpha^q)(t)N(t+\tau)}\right)_P,\end{aligned} \tag{4.100a}$$

where we have introduced the notation

$$\left(\overline{(\alpha^{*p}\alpha^q)(t)N(t+\tau)}\right)_P \equiv \int d^2\alpha \int d^2\alpha_0\, \alpha_0^{*p}\alpha_0^q N(\alpha,\alpha^*)P(\alpha,\alpha^*,t+\tau;\alpha_0,\alpha_0^*,t), \tag{4.100b}$$

and

$$P(\alpha,\alpha^*,t+\tau;\alpha_0,\alpha_0^*,t) = P(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0)P(\alpha_0,\alpha_0^*,t) \tag{4.101}$$

is the two-time, or joint, distribution. Thus, the correspondence with a classical statistical description has been extended one step further. Equation (4.100b) is formally equivalent to the formula for calculating two-time averages in a classical statistical theory.

4.3.4 More General Two-Time Averages Using the P Representation

We have seen that antinormal-ordered one-time averages can be calculated using the P representation [Sect. 4.1.3]; although, with some inconvenience, since the expressions for these averages involve derivatives of the P distribution. The situation is similar when we consider two-time averages that are not in normal-ordered time-ordered form. To see how (4.100) must be modified to give these averages we will seek a phase-space expression using the P representation for the general average ($\tau \geq 0$)

$$\langle \hat{O}_{r,q,m}(t)\hat{N}(t+\tau)\hat{O}^\dagger_{s,p,n}(t)\rangle = \mathrm{tr}\left\{\left(e^{\mathcal{L}\tau}[\hat{O}^\dagger_{s,p,n}\rho(t)\hat{O}_{r,q,m}]\right)\hat{N}\right\}, \tag{4.102}$$

where

$$\hat{O}_{k_1,k_2,k_3} \equiv a^{\dagger k_1}a^{k_2}a^{\dagger k_3}, \tag{4.103}$$

and $\hat{N}$ is again the arbitrary normal-ordered operator defined by the series expansion (4.79). Once we have a solution to this problem, results for various combinations of normal-ordered and antinormal-ordered operators will follow with little extra effort.

We begin as before, using (4.83) and (4.94) to write

$$\begin{aligned}\langle \hat{O}_{r,q,m}(t)\hat{N}(t+\tau)\hat{O}^\dagger_{s,p,n}(t)\rangle &= \frac{1}{\pi}\int d^2\alpha \left[e^{L^{(a)}(\alpha,\alpha^*,\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\alpha^*})\tau} F^{(a)}_{\hat{O}^\dagger_{s,p,n}\rho(t)\hat{O}_{r,q,m}}(\alpha,\alpha^*)\right] F^{(n)}_{\hat{N}}(\alpha,\alpha^*)\\ &= \frac{1}{\pi}\int d^2\alpha \left[e^{L^{(a)}(\alpha,\alpha^*,\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\alpha^*})\tau} \frac{1}{\pi^2}\int d^2z\, \tilde{F}^{(a)}_{\hat{O}^\dagger_{s,p,n}\rho(t)\hat{O}_{r,q,m}}(z,z^*)\right.\\ &\qquad \left.\times e^{-iz^*\alpha^*}e^{-iz\alpha}\right] F^{(n)}_{\hat{N}}(\alpha,\alpha^*);\end{aligned} \tag{4.104}$$

the second line follows from (4.60). Our aim now is to express the function $\tilde{F}^{(a)}_{\hat{O}^\dagger_{s,p,n}\rho(t)\hat{O}_{r,q,m}}(z,z^*)$ in terms of $\tilde{F}^{(a)}_{\rho(t)}(z,z^*)$ and its derivatives. Using (4.59) and (4.103), we have

$$\begin{aligned}\tilde{F}^{(a)}_{\hat{O}^\dagger_{s,p,n}\rho(t)\hat{O}_{r,q,m}}(z,z^*) &= \pi\mathrm{tr}\left[a^n a^{\dagger p}a^s\rho(t)a^{\dagger r}a^q a^{\dagger m}e^{iz^*a^\dagger}e^{iza}\right]\\ &= \pi\mathrm{tr}\left[\rho(t)a^{\dagger r}a^q a^{\dagger m}e^{iz^*a^\dagger}e^{iza}a^n a^{\dagger p}a^s\right]\\ &= \frac{\partial^{m+n}}{\partial(iz^*)^m\partial(iz)^n}\pi\mathrm{tr}\left[\rho(t)a^{\dagger r}a^q e^{iz^*a^\dagger}e^{iza}a^{\dagger p}a^s\right],\end{aligned}$$

and then, from (3.78),

$$
\begin{aligned}
&\tilde{F}^{(a)}_{\hat{O}^\dagger_{n,p,s}\rho(t)\hat{O}_{r,q,m}}(z,z^*)\\
&= \frac{\partial^{m+n}}{\partial(iz^*)^m\partial(iz)^n}\pi\mathrm{tr}\big[\rho(t)a^{\dagger r}e^{iz^*a^\dagger}e^{iza}(a+iz^*)^q a^{\dagger p}a^s\big]\\
&= \frac{\partial^{m+n}}{\partial(iz^*)^m\partial(iz)^n}\left(\frac{\partial}{\partial(iz)}+iz^*\right)^q\pi\mathrm{tr}\big[\rho(t)a^{\dagger r}e^{iz^*a^\dagger}e^{iza}a^{\dagger p}a^s\big]\\
&= \frac{\partial^{m+n}}{\partial(iz^*)^m\partial(iz)^n}\left(\frac{\partial}{\partial(iz)}+iz^*\right)^q\pi\mathrm{tr}\big[\rho(t)a^{\dagger r}(a^\dagger+iz)^p e^{iz^*a^\dagger}e^{iza}a^s\big]\\
&= \frac{\partial^{m+n}}{\partial(iz^*)^m\partial(iz)^n}\left(\frac{\partial}{\partial(iz)}+iz^*\right)^q\left(\frac{\partial}{\partial(iz^*)}+iz\right)^p\pi\mathrm{tr}\big[\rho(t)a^{\dagger r}e^{iz^*a^\dagger}e^{iza}a^s\big]\\
&= \frac{\partial^{m+n}}{\partial(iz^*)^m\partial(iz)^n}\left(\frac{\partial}{\partial(iz)}+iz^*\right)^q\left(\frac{\partial}{\partial(iz^*)}+iz\right)^p\\
&\quad\times\frac{\partial^{r+s}}{\partial(iz^*)^r\partial(iz)^s}\tilde{F}^{(a)}_{\rho(t)}(z,z^*).
\end{aligned}
$$

We write this to reflect the order of the operators in (4.103):

$$
\begin{aligned}
\tilde{F}^{(a)}_{\hat{O}^\dagger_{n,p,s}\rho(t)\hat{O}_{r,q,m}}(z,z^*) &= \frac{\partial^m}{\partial(iz^*)^m}\left(\frac{\partial}{\partial(iz)}+iz^*\right)^q\frac{\partial^r}{\partial(iz^*)^r}\\
&\quad\times\frac{\partial^n}{\partial(iz)^n}\left(\frac{\partial}{\partial(iz^*)}+iz\right)^p\frac{\partial^s}{\partial(iz)^s}\tilde{F}^{(a)}_{\rho(t)}(z,z^*).
\end{aligned}
\tag{4.105}
$$

We now substitute the Fourier transform of $F^{(a)}_{\rho(t)}(\alpha,\alpha^*)$ for $\tilde{F}^{(a)}_{\rho(t)}(z,z^*)$ to obtain

$$
\begin{aligned}
&\tilde{F}^{(a)}_{\hat{O}^\dagger_{n,p,s}\rho(t)\hat{O}_{r,q,m}}(z,z^*)\\
&= \frac{\partial^m}{\partial(iz^*)^m}\left(\frac{\partial}{\partial(iz)}+iz^*\right)^q\frac{\partial^r}{\partial(iz^*)^r}\frac{\partial^n}{\partial(iz)^n}\left(\frac{\partial}{\partial(iz^*)}+iz\right)^p\frac{\partial^s}{\partial(iz)^s}\\
&\quad\times\int d^2\lambda\, F^{(a)}_{\rho(t)}(\lambda,\lambda^*)e^{iz^*\lambda^*}e^{iz\lambda}\\
&= \int d^2\lambda\, F^{(a)}_{\rho(t)}(\lambda,\lambda^*)\lambda^s\left(\lambda^*+\frac{\partial}{\partial\lambda}\right)^p\lambda^n\lambda^{*r}\left(\lambda+\frac{\partial}{\partial\lambda^*}\right)^q\lambda^{*m}e^{iz^*\lambda^*}e^{iz\lambda}\\
&= \int d^2\lambda\left[\lambda^{*m}\left(\lambda-\frac{\partial}{\partial\lambda^*}\right)^q\lambda^{*r}\lambda^n\left(\lambda^*-\frac{\partial}{\partial\lambda}\right)^p\lambda^s F^{(a)}_{\rho(t)}(\lambda,\lambda^*)\right]e^{iz^*\lambda^*}e^{iz\lambda},
\end{aligned}
\tag{4.106}
$$

where the last line follows after repeated integration by parts. When we use this result in (4.104) the integral with respect to z gives a δ-function, $\delta^{(2)}(\alpha-\lambda)$, and the integral with respect to λ is then trivially performed; we find ($\tau\geq 0$)

$$
\begin{aligned}
&\langle\hat{O}_{r,q,m}(t)\hat{N}(t+\tau)\hat{O}^\dagger_{n,p,s}(t)\rangle\\
&\quad= \frac{1}{\pi}\int d^2\alpha\left[e^{L^{(a)}\left(\alpha,\alpha^*,\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\alpha^*}\right)\tau}\alpha^{*m}\left(\alpha-\frac{\partial}{\partial\alpha^*}\right)^q\alpha^{*r}\right.\\
&\qquad\left.\times\,\alpha^n\left(\alpha^*-\frac{\partial}{\partial\alpha}\right)^p\alpha^s F^{(a)}_{\rho(t)}(\alpha,\alpha^*)\right]F^{(n)}_{\hat{N}}(\alpha,\alpha^*).
\end{aligned}
\tag{4.107}
$$

If we proceed, as below (4.98), to express this result in terms of $P(\alpha_0,\alpha_0^*,t)$ and $P(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0)$, (4.107) becomes ($\tau\geq 0$)

$$
\begin{aligned}
&\langle\hat{O}_{r,q,m}(t)\hat{N}(t+\tau)\hat{O}^\dagger_{n,p,s}(t)\rangle\\
&\quad= \int d^2\alpha\int d^2\alpha_0\, N(\alpha,\alpha^*)P(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0)\\
&\qquad\times\alpha_0^{*m}\left(\alpha_0-\frac{\partial}{\partial\alpha_0^*}\right)^q\alpha_0^{*r}\alpha_0{}^n\left(\alpha_0^*-\frac{\partial}{\partial\alpha_0}\right)^p\alpha_0{}^s P(\alpha_0,\alpha_0^*,t).
\end{aligned}
\tag{4.108}
$$

The replacement of $a^{\dagger p}$ and a^q by differential operators, below (4.104), may also be performed in the reverse order; this gives an alternative to (4.108) in the form ($\tau\geq 0$)

$$
\begin{aligned}
&\langle\hat{O}_{r,q,m}(t)\hat{N}(t+\tau)\hat{O}^\dagger_{n,p,s}(t)\rangle\\
&\quad= \int d^2\alpha\int d^2\alpha_0\, N(\alpha,\alpha^*)P(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0)\\
&\qquad\times\alpha_0{}^n\left(\alpha_0^*-\frac{\partial}{\partial\alpha_0}\right)^p\alpha_0{}^s\alpha_0^{*m}\left(\alpha_0-\frac{\partial}{\partial\alpha_0^*}\right)^q\alpha_0^{*r}P(\alpha_0,\alpha_0^*,t).
\end{aligned}
\tag{4.109}
$$

With $p=q=0$, both of these expressions reproduce the result (4.100) for the average $\langle a^{\dagger m+r}(t)\hat{N}(t+\tau)a^{n+s}(t)\rangle$. When $p\neq 0$, or $q\neq 0$, derivatives of $P(\alpha_0,\alpha_0^*,t)$ are involved, as in (4.23). Equation (4.23a) can be recovered from either (4.108) or (4.109); for example, with $q\neq 0$, $\hat{N}=a^{\dagger p}$, $\tau=0$, and $r=m=n=p=s=0$. Similarly, (4.23b) can be recovered with $p\neq 0$, $\hat{N}=a^q$, $\tau=0$, and $r=q=m=n=s=0$. There are other combinations of parameters that also recover these earlier results.

A number of results for two-time averages of operators expressed as normal-ordered and antinormal-ordered series now follow from (4.108) and (4.109). We introduce the normal-ordered series

$$
\hat{N}_1 = N_1(a,a^\dagger)\equiv\sum_{p,q}C^{(n)}_{1p,q}a^{\dagger p}a^q,
\tag{4.110a}
$$

$$
\hat{N}_2 = N_2(a,a^\dagger)\equiv\sum_{p,q}C^{(n)}_{2pq}a^{\dagger p}a^q,
\tag{4.110b}
$$

and the antinormal-ordered series

$$\hat{A}_1 = A_1(a,a^\dagger) \equiv \sum_{p,q} C^{(a)}_{1pq} a^q a^{\dagger p}, \tag{4.111a}$$

$$\hat{A}_2 = A_2(a,a^\dagger) \equiv \sum_{p,q} C^{(a)}_{2pq} a^q a^{\dagger p}. \tag{4.111b}$$

Then, applying (4.108) term by term, we prove the following ($\tau \geq 0$):

$$\begin{aligned}
&\langle \hat{N}_1(t)\hat{N}(t+\tau)\hat{N}_2(t)\rangle \\
&\quad = \int d^2\alpha \int d^2\alpha_0\, N(\alpha,\alpha^*)P(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0) \\
&\qquad \times \overleftarrow{N}_1\Big(\alpha_0 - \frac{\partial}{\partial\alpha_0^*},\alpha_0^*\Big)\overleftarrow{N}_2\Big(\alpha_0,\alpha_0^* - \frac{\partial}{\partial\alpha_0}\Big)P(\alpha_0,\alpha_0^*,t),
\end{aligned} \tag{4.112a}$$

$$\begin{aligned}
&\langle \hat{N}_1(t)\hat{N}(t+\tau)\hat{A}_2(t)\rangle \\
&\quad = \int d^2\alpha \int d^2\alpha_0\, N(\alpha,\alpha^*)P(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0) \\
&\qquad \times \overleftarrow{N}_1\Big(\alpha_0 - \frac{\partial}{\partial\alpha_0^*},\alpha_0^*\Big)\overrightarrow{A}_2\Big(\alpha_0,\alpha_0^* - \frac{\partial}{\partial\alpha_0}\Big)P(\alpha_0,\alpha_0^*,t),
\end{aligned} \tag{4.112b}$$

$$\begin{aligned}
&\langle \hat{A}_1(t)\hat{N}(t+\tau)\hat{N}_2(t)\rangle \\
&\quad = \int d^2\alpha \int d^2\alpha_0\, N(\alpha,\alpha^*)P(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0) \\
&\qquad \times \overrightarrow{A}_1\Big(\alpha_0 - \frac{\partial}{\partial\alpha_0^*},\alpha_0^*\Big)\overleftarrow{N}_2\Big(\alpha_0,\alpha_0^* - \frac{\partial}{\partial\alpha_0}\Big)P(\alpha_0,\alpha_0^*,t),
\end{aligned} \tag{4.112c}$$

$$\begin{aligned}
&\langle \hat{A}_1(t)\hat{N}(t+\tau)\hat{A}_2(t)\rangle \\
&\quad = \int d^2\alpha \int d^2\alpha_0\, N(\alpha,\alpha^*)P(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0) \\
&\qquad \times \overrightarrow{A}_1\Big(\alpha_0 - \frac{\partial}{\partial\alpha_0^*},\alpha_0^*\Big)\overrightarrow{A}_2\Big(\alpha_0,\alpha_0^* - \frac{\partial}{\partial\alpha_0}\Big)P(\alpha_0,\alpha_0^*,t).
\end{aligned} \tag{4.112d}$$

The arrows indicate whether the power series are to be written with the differential operators placed to the right or to the left. Equation (4.109) allows the order of the functions N_1, N_2, A_1, and A_2 to be reversed in these expressions.

Note 4.6 We have not exhausted all combinations of normal-ordered and antinormal-ordered operators here. If $\hat{N}$ is replaced by an antinormal-ordered series [Eq. (4.64)], it can be shown that $N(\alpha,\alpha^*)$ may be replaced in (4.112a)–(4.112d) by either $\overrightarrow{A}(\alpha-\frac{\partial}{\partial\alpha^*},\alpha^*)$ or $\overrightarrow{A}(\alpha,\alpha^*-\frac{\partial}{\partial\alpha})$. The resulting expressions reproduce (4.23a) and (4.23b), respectively, when $\hat{N}_1 = \hat{N}_2 = \hat{A}_1 = \hat{A}_2 = 1$ and $\hat{A} = a^q a^{\dagger p}$. To prove this, use the relationship between $F^{(n)}_{\hat{A}}(\alpha,\alpha^*)$ and $F^{(a)}_{\hat{A}}(\alpha,\alpha^*)$ given by (4.78).

4.3.5 Two-Time Averages Using the Q and Wigner Representations

Just as the operator averages corresponding to the moments of the single-time distribution vary from one representation to the other, so too do the averages corresponding to the moments of the two-time, or joint, distribution. In the Q representation a calculation parallel to that of Sect. 4.3.3 shows that antinormal-ordered, reverse-time-ordered, two-time averages are given by ($\tau \geq 0$)

$$\langle a^q(t)\hat{A}(t+\tau)a^{\dagger p}(t)\rangle = \Big(\overline{(\alpha^{*p}\alpha^q)(t)A(t+\tau)}\Big)_Q, \tag{4.113a}$$

with

$$\begin{aligned}
&\Big(\overline{(\alpha^{*p}\alpha^q)(t)A(t+\tau)}\Big)_Q \\
&\qquad \equiv \int d^2\alpha \int d^2\alpha_0\, \alpha_0^{*p}\alpha_0^q A(\alpha,\alpha^*)Q(\alpha,\alpha^*,t+\tau;\alpha_0,\alpha_0^*,t),
\end{aligned} \tag{4.113b}$$

and

$$Q(\alpha,\alpha^*,t+\tau;\alpha_0,\alpha_0^*,t) = Q(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0)Q(\alpha_0,\alpha_0^*,t), \tag{4.114}$$

where $\hat{A}$ is any operator written as a series in antinormal order [Eq. (4.64)]. More general averages not of the antinormal-ordered, reverse-time-ordered form involve derivatives of the Q distribution after the fashion of (4.112a)–(4.112d).

Exercise 4.6 Show that ($\tau \geq 0$)

$$\begin{aligned}
&\langle \hat{A}_1(t)\hat{A}(t+\tau)\hat{A}_2(t)\rangle \\
&\quad = \int d^2\alpha \int d^2\alpha_0\, A(\alpha,\alpha^*)Q(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0) \\
&\qquad \times \overleftarrow{A}_1\Big(\alpha_0,\alpha_0^* + \frac{\partial}{\partial\alpha_0}\Big)\overleftarrow{A}_2\Big(\alpha_0 + \frac{\partial}{\partial\alpha_0^*},\alpha_0^*\Big)Q(\alpha_0,\alpha_0^*,t),
\end{aligned} \tag{4.115a}$$

$$\begin{aligned}
&\langle \hat{A}_1(t)\hat{A}(t+\tau)\hat{N}_2(t)\rangle \\
&\quad = \int d^2\alpha \int d^2\alpha_0\, A(\alpha,\alpha^*)Q(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0) \\
&\qquad \times \overleftarrow{A}_1\Big(\alpha_0,\alpha_0^* + \frac{\partial}{\partial\alpha_0}\Big)\overrightarrow{N}_2\Big(\alpha_0 + \frac{\partial}{\partial\alpha_0^*},\alpha_0^*\Big)Q(\alpha_0,\alpha_0^*,t),
\end{aligned} \tag{4.115b}$$

$$\begin{aligned}\langle\hat N_1(t)\hat A(t+\tau)\hat A_2(t)\rangle &= \int d^2\alpha\int d^2\alpha_0\, A(\alpha,\alpha^*)Q(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0)\\ &\quad\times\overrightarrow{N_1}\Big(\alpha_0,\alpha_0^*+\frac{\partial}{\partial\alpha_0}\Big)\overleftarrow{A_2}\Big(\alpha_0+\frac{\partial}{\partial\alpha_0^*},\alpha_0^*\Big)Q(\alpha_0,\alpha_0^*,t),\end{aligned}\tag{4.115c}$$

$$\begin{aligned}\langle\hat N_1(t)\hat A(t+\tau)\hat N_2(t)\rangle &= \int d^2\alpha\int d^2\alpha_0\, A(\alpha,\alpha^*)Q(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0)\\ &\quad\times\overrightarrow{N_1}\Big(\alpha_0,\alpha_0^*+\frac{\partial}{\partial\alpha_0}\Big)\overrightarrow{N_2}\Big(\alpha_0+\frac{\partial}{\partial\alpha_0^*},\alpha_0^*\Big)Q(\alpha_0,\alpha_0^*,t).\end{aligned}\tag{4.115d}$$

As mentioned in Note 4.6, if $\hat A$ is replaced by an operator $\hat N = N(a,a^\dagger)$ written as a normal-ordered series, $A(\alpha,\alpha^*)$ may be replaced in these expressions by either $\overrightarrow{N}\big(\alpha+\frac{\partial}{\partial\alpha^*},\alpha^*\big)$ or $\overrightarrow{N}\big(\alpha,\alpha^*+\frac{\partial}{\partial\alpha}\big)$. From the resulting expressions we can recover (4.24a) and (4.24b) by setting $\hat A_1=\hat A_2=\hat N_1=\hat N_2=1$ and $N(a,a^\dagger)=a^{\dagger p}a^q$.

We might expect the operator averages that correspond to moments of the two-time distribution in the Wigner representation to be some rather tangled mess. The symmetric-ordered operators related to moments of the one-time distribution are themselves a little imposing beyond the first few orders; how must we distribute the "t's" and "$t+\tau$'s" within the terms of the symmetric operator sums [Eqs. (4.29)] to come up with the two-time operator whose average is given by a double integration like (4.100) or (4.113)? The answer to this question is found by studying Sect. 4.3.3 a little more carefully to find out what really makes the calculation there work. Needless to say, the extension of this calculation to two-time averages calculated in the Wigner representation is going to call for a little more algebraic muscle.

First, note that a sum of averages ($\tau\ge 0$)

$$\sum_{i,j}\langle\hat O_i(t)\hat S(t+\tau)\hat O_j(t)\rangle = \sum_{i,j}\mathrm{tr}\big\{\big(e^{\mathcal{L}\tau}[\hat O_j\rho(t)\hat O_i]\big)\hat S\big\}\tag{4.116}$$

can be written as a phase-space integral analogous to (4.96):

$$\begin{aligned}&\sum_{i,j}\langle\hat O_i(t)\hat S(t+\tau)\hat O_j(t)\rangle\\ &\quad=\frac{1}{\pi}\int d^2\alpha\Big[e^{L^{(s)}(\alpha,\alpha^*,\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\alpha^*})\tau}\sum_{i,j}F^{(s)}_{\hat O_j\rho(t)\hat O_i}(\alpha,\alpha^*)\Big]F^{(s)}_{\hat S}(\alpha,\alpha^*),\end{aligned}\tag{4.117}$$

where we have used (4.84) and (4.94), and $\hat S$ denotes any operator written as a symmetric-ordered series [Eq. (4.81)]. Now, the point on which the calculation of Sect. 4.3.3 turns is found in the fourth line of the equation below (4.96); if we can substitute $\tilde F^{(s)}_{\rho(t)}(z,z^*)$ for $\tilde F^{(a)}_{\rho(t)}(z,z^*)$ here we will be able to proceed in a parallel calculation to a result analogous to (4.100) – with W replacing P, and S replacing N. But to connect such a calculation with (4.117) we must answer one question: What operators $\hat O_i$ and $\hat O_j$ must be chosen so that

$$\sum_{i,j}\tilde F^{(s)}_{\hat O_j\rho(t)\hat O_i}(z,z^*) = \frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\tilde F^{(s)}_{\rho(t)}(z,z^*)\,?$$

With the answer to this question the two-time operator average obtained from moments of the two-time distribution in the Wigner representation will be the average (4.116).

The key to an answer lies with the following observation. Using (4.71) and the Baker-Hausdorff theorem [Eq. (4.8)], we find

$$\begin{aligned}\frac{\partial}{\partial(iz)}\tilde F^{(s)}_{\rho(t)}(z,z^*) &= \frac{\partial}{\partial(iz)}\pi\mathrm{tr}\big[\rho(t)e^{iz^*a^\dagger+iza}\big]\\ &= \frac{\partial}{\partial(iz)}\tfrac12\pi\mathrm{tr}\big[\rho(t)\big(e^{\frac12|z|^2}e^{iza}e^{iz^*a^\dagger}+e^{-\frac12|z|^2}e^{iz^*a^\dagger}e^{iza}\big)\big]\\ &= \tfrac12\pi\mathrm{tr}\big\{\rho(t)\big[\big(a-\tfrac12 iz^*\big)e^{iz^*a^\dagger+iza}+e^{iz^*a^\dagger+iza}\big(a+\tfrac12 iz^*\big)\big]\big\}\\ &= \tfrac12\big[\tilde F^{(s)}_{a\rho(t)}(z,z^*)+\tilde F^{(s)}_{\rho(t)a}(z,z^*)\big],\end{aligned}\tag{4.118a}$$

and, in a similar fashion,

$$\frac{\partial}{\partial(iz^*)}\tilde F^{(s)}_{\rho(t)}(z,z^*) = \tfrac12\big[\tilde F^{(s)}_{a^\dagger\rho(t)}(z,z^*)+\tilde F^{(s)}_{\rho(t)a^\dagger}(z,z^*)\big].\tag{4.118b}$$

Also, if we wish to obtain an answer in a form that preserves the relationship to operators written in symmetric order, we must order the differential operators appearing in (4.118) in a corresponding fashion. Thus, we write

$$\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q} = \left(\frac{\partial^p}{\partial(iz^*)^p}\frac{\partial^q}{\partial(iz)^q}\right)_S,\tag{4.119}$$

where the right-hand side is the average of the $(p+q)!/(p!q!)$ orderings of the p differential operators $\partial/\partial(iz^*)$ and the q differential operators $\partial/\partial(iz)$. Now the answer to our question is accessible. To reach it, however, still requires a little combinatorics. The final step is left as an exercise:

Exercise 4.7 Use (4.118a), (4.118b), and (4.119) to show that

$$\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\tilde{F}^{(s)}_{\rho(t)}(z,z^*) = \frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\tilde{F}^{(s)}_{(a^{\dagger p}:\rho(t):a^q)^{(k)}_S}(z,z^*), \tag{4.120}$$

with

$$\left(a^{\dagger p}:\rho(t):a^q\right)^{(k)}_S \equiv \frac{p!q!}{(p+q)!}\sum_{\{\hat{O}_j\}}\hat{O}_{p+q}\hat{O}_{p+q-1}\cdots\hat{O}_{k+1}\rho(t)\hat{O}_k\cdots\hat{O}_1, \tag{4.121}$$

where the summation in (4.121) is taken over all different permutations $\hat{O}_1\cdots\hat{O}_{p+q}$ of p creation operators and q annihilation operators – i.e. $\rho(t)$ is placed into each term of $\left(a^{\dagger p}a^q\right)_S$ k places from the extreme right.

Equation (4.120) now allows us to follow the steps that led to (4.97) to obtain the corresponding result

$$\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}F^{(s)}_{(a^{\dagger p}:\rho(t):a^q)^{(k)}_S}(\alpha,\alpha^*) = F^{(s)}_{\rho(t)}(\alpha,\alpha^*)\alpha^{*p}\alpha^q. \tag{4.122}$$

The series of operators $\hat{O}_i$ and $\hat{O}_j$ appearing in (4.117) must now be chosen to connect with this result. The choice is fairly obvious from the associated function that appears on the left-hand side of (4.122); we have

$$\begin{aligned}
&\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\left\langle\left(a^{\dagger p}(t):\hat{S}(t+\tau):a^q(t)\right)^{(k)}_S\right\rangle\\
&\quad=\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\frac{p!q!}{(p+q)!}\\
&\qquad\times\sum_{\{\hat{O}_j\}}\langle\hat{O}_{p+q}(t)\cdots\hat{O}_{k+1}(t)\hat{S}(t+\tau)\hat{O}_k(t)\cdots\hat{O}_1(t)\rangle\\
&\quad=\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\frac{p!q!}{(p+q)!}\\
&\qquad\times\sum_{\{\hat{O}_j\}}\mathrm{tr}\left\{\left(e^{\mathcal{L}\tau}[\hat{O}_k\cdots\hat{O}_1\rho(t)\hat{O}_{p+q}\cdots\hat{O}_{k+1}]\right)\hat{S}\right\},
\end{aligned}$$

where we have used (1.102). The order of the subscripts in the sum over permutations of the operator product $a^{\dagger p}a^q$ can be changed with no effect, since operator sequences in every order are covered in the sum. Then

$$\begin{aligned}
&\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\left\langle\left(a^{\dagger p}(t):\hat{S}(t+\tau):a^q(t)\right)^{(k)}_S\right\rangle\\
&\quad=\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\frac{p!q!}{(p+q)!}\\
&\qquad\times\sum_{\{\hat{O}_j\}}\mathrm{tr}\left\{\left(e^{\mathcal{L}\tau}[\hat{O}_{p+q}\cdots\hat{O}_{p+q-k+1}\rho(t)\hat{O}_{p+q-k}\cdots\hat{O}_1]\right)\hat{S}\right\}.
\end{aligned}$$

In the operator sequences on the right-hand side of this expression $\rho(t)$ is inserted k places from the extreme *left*, in contrast to its position k places from the extreme *right* in the definition (4.121). This difference is removed, however, by a change of summation index, with $p+q-k\to k$; after making this change we arrive at the desired explicit form for (4.117); using (4.84) and (4.94):

$$\begin{aligned}
&\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\left\langle\left(a^{\dagger p}(t):\hat{S}(t+\tau):a^q(t)\right)^{(k)}_S\right\rangle\\
&\quad=\mathrm{tr}\left\{\left(e^{\mathcal{L}\tau}\left[\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\left(a^{\dagger p}:\rho(t):a^q\right)^{(k)}_S\right]\right)\hat{S}\right\}\\
&\quad=\frac{1}{\pi}\int d^2\alpha\left[e^{L^{(s)}(\alpha,\alpha^*,\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\alpha^*})\tau}\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\right.\\
&\qquad\left.\times F^{(s)}_{(a^{\dagger p}:\rho(t):a^q)^{(k)}_S}(\alpha,\alpha^*)\right]F^{(s)}_{\hat{S}}(\alpha,\alpha^*).
\end{aligned} \tag{4.123}$$

Equations (4.122) and (4.123) allow the two-time operator average on the left-hand side of (4.123) to be calculated as a phase-space average with respect to the two-time Wigner distribution. Following the steps leading from (4.98) to (4.100) we obtain the corresponding result ($\tau\ge 0$)

$$\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\left\langle\left(a^{\dagger p}(t):\hat{S}(t+\tau):a^q(t)\right)^{(k)}_S\right\rangle = \left(\overline{(\alpha^{*p}\alpha^q)(t)S(t+\tau)}\right)_W, \tag{4.124a}$$

with

$$\begin{aligned}
&\left(\overline{(\alpha^{*p}\alpha^q)(t)S(t+\tau)}\right)_W\\
&\quad\equiv\int d^2\alpha\int d^2\alpha_0\,\alpha_0^{*p}\alpha_0^qS(\alpha,\alpha^*)W(\alpha,\alpha^*,t+\tau;\alpha_0,\alpha_0^*,t),
\end{aligned} \tag{4.124b}$$

and

$$W(\alpha,\alpha^*,t+\tau;\alpha_0,\alpha_0^*,t) = W(\alpha,\alpha^*,\tau|\alpha_0,\alpha_0^*,0)W(\alpha_0,\alpha_0^*,t). \tag{4.125}$$

We have again managed to construct a relationship between ordered operator two-time averages and two-time averages in the corresponding "classical" statistical system. However, the sum of operator averages appearing on the left-hand side of (4.124a) makes this a rather more formidable relationship than the corresponding relationships for the P and Q representations [Eqs. (4.100) and (4.113)].

To convince ourselves of the consistency of our result we should perhaps show that (4.124) is able to reproduce the expression for calculating one-time averages in the Wigner representation [Eq. (4.31)]. This is clear when we specialize to one-time averages by either taking $p=q=0$, or $\hat{S}=1$; in both cases we need only observe that

$$\sum_{k=0}^{p+q}\binom{p+q}{k} = (1+1)^{p+q} = 2^{p+q}.$$

It is less obvious, however, that the single-time result is recovered when τ is set to zero. Then (4.124) becomes

$$\begin{aligned}\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\Big\langle\big(a^{\dagger p}(t){:}\hat{S}(t){:}a^q(t)\big)_S^{(k)}\Big\rangle \\ = \int d^2\alpha\,\alpha^{*p}\alpha^q S(\alpha,\alpha^*)W(\alpha,\alpha^*,t).\end{aligned}$$

If this is to correspond to (4.31), the phase-space function

$$\alpha^{*p}\alpha^q S(\alpha,\alpha^*) = \sum_{p',q'} C^{(s)}_{p',q'}\alpha^{*p+p'}\alpha^{q+q'}$$

that appears with the Wigner distribution in the integrand on the right-hand side must be the symmetric-ordered associated function for the operator that appears on the left-hand side – i.e. for the operator

$$\begin{aligned}\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\Big\langle\big(a^{\dagger p}{:}\hat{S}{:}a^q\big)_S^{(k)}\Big\rangle \\ = \sum_{p',q'} C^{(s)}_{p',q'}\left[\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\big(a^{\dagger p}{:}\,(a^{\dagger p'}a^{q'})_S\,{:}a^q\big)_S^{(k)}\right].\end{aligned}$$

We know that $\big(a^{\dagger p+p'}a^{q+q'}\big)_S$ is the operator with the symmetric-ordered associated function $\alpha^{*p+p'}\alpha^{q+q'}$; thus, we must show that

$$\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\big(a^{\dagger p}{:}\,(a^{\dagger p'}a^{q'})_S\,{:}a^q\big)_S^{(k)} = \big(a^{\dagger p+p'}a^{q+q'}\big)_S. \tag{4.126}$$

The proof is constructed by using the identity (4.28) to write

$$\begin{aligned}\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\big(a^{\dagger p}{:}\,(a^{\dagger p'}a^{q'})_S\,{:}a^q\big)_S^{(k)} \\ = \frac{\partial^{p'+q'}}{\partial(iz^*)^{p'}\partial(iz)^{q'}}\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\big(a^{\dagger p}{:}e^{iz^*a^\dagger+iza}{:}a^q\big)_S^{(k)}\bigg|_{z=z^*=0}.\end{aligned}$$

Then, using

$$\frac{\partial}{\partial(iz)}e^{iz^*a^\dagger+iza} = \tfrac{1}{2}\big(ae^{iz^*a^\dagger+iza}+e^{iz^*a^\dagger+iza}a\big), \tag{4.127a}$$

$$\frac{\partial}{\partial(iz^*)}e^{iz^*a^\dagger+iza} = \tfrac{1}{2}\big(a^\dagger e^{iz^*a^\dagger+iza}+e^{iz^*a^\dagger+iza}a^\dagger\big), \tag{4.127b}$$

a calculation parallel to the one leading from (4.118) to (4.120) gives

$$\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\big(a^{\dagger p}{:}e^{iz^*a^\dagger+iza}{:}a^q\big)_S^{(k)} = \frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}e^{iz^*a^\dagger+iza}. \tag{4.128}$$

Substituting this result and making a second use of (4.28), we have

$$\begin{aligned}\frac{1}{2^{p+q}}\sum_{k=0}^{p+q}\binom{p+q}{k}\big(a^{\dagger p}{:}\,(a^{\dagger p'}a^{q'})_S\,{:}a^q\big)_S^{(k)} \\ = \frac{\partial^{p+p'+q+q'}}{\partial(iz^*)^{p+p'}\partial(iz)^{q+q'}}e^{iz^*a^\dagger+iza}\bigg|_{z=z^*=0} \\ = \big(a^{\dagger p+p'}a^{q+q'}\big)_S.\end{aligned}$$

It is possible to derive more general expressions for two-time averages in the Wigner representation – expressions that involve partial derivatives, after the fashion of the results (4.112) and (4.115) for the P and Q representations. We have no use, however, for these expressions later in the book and therefore we will not bother with their derivation here. In general we are interested only in the simple relationships (4.100), (4.113), and (4.124), where two-time operator averages are given by moments of the two-time phase-space distributions. It is important to realize, however, that within each of the three representations we have discussed many two-time averages simply cannot be calculated in terms of a simple "classical" integral; the more complicated expressions such as (4.112) and (4.115) are needed when the ordering is inappropriate for the chosen representation. When calculating single-time averages we always have the option of reordering the operators to suit the representation. Thus, $\langle a^\dagger a\rangle$ can be calculated as $\langle a^\dagger a\rangle = \left(\overline{\alpha^*\alpha}\right)_P$ in the P representation, $\langle aa^\dagger\rangle - 1 = \left(\overline{\alpha^*\alpha}\right)_Q - 1$ in the Q representation, or as $\frac{1}{2}\left(\langle a^\dagger a\rangle + \langle aa^\dagger\rangle\right) - \frac{1}{2} = \left(\overline{\alpha^*\alpha}\right)_W - \frac{1}{2}$ in the Wigner representation. On the other hand, while an average like $\langle a^\dagger(t+\tau)a(t)\rangle$, or $\langle a(t+\tau)a(t)\rangle$, can be calculated as a "classical" integral in the P representation [Eq. (4.100)], we generally do not have commutation relations to tell us how to reorder the operators so that the same result can be obtained as simply in either the Q or the Wigner representations. Applications in quantum optics are ultimately concerned with the normal-ordered time-ordered averages that arise in the theory of photodetection [4.11, 4.12]. Our phase-space results for two-time (more generally multi-time) averages clearly distinguishes the P representation as the most suited to the treatment of problems in quantum optics – results for multi-time averages show this even more clearly than do results for one-time averages.

Note 4.7 The assertion that the P representation is the most suited to problems in quantum optics perhaps requires some qualification. The P representation gains its special status from the theory of photoelectric detection, in which normal-ordered time-ordered averages appear. Therefore questions that are related in an immediate way to the ultimate observation of photons through the photoelectric effect lead in a natural way to a phase-space formulation in terms of the P representation. But there are questions of interest which need not be stated in terms of the photoelectric emission that ultimately completes a measurement process. Certainly then, there are situations in which, as a mathematical tool, the Q or the Wigner representation might be preferred over the P representation. An important consideration in this regard is the fact that the P distribution may be a generalized function. If this is so we do not gain much physical insight, and probably little mathematical assistance, by using the P representation. On the other hand, the Q and Wigner distributions are always well-behaved functions (although the Wigner distribution may take on negative values). For this reason the Q or Wigner representation is often the choice for studies of nonclassical states of the electromagnetic field – for example, squeezed states, in one sense, are related most directly to the Wigner representation.

Having said this, it is still important to reiterate the observation above concerning multi-time averages. When we use a phase-space representation to convert an operator master equation into a Fokker–Planck equation, we do not merely set up a representation for some state of the electromagnetic field; we set up a correspondence between quantum and classical processes *that evolve in time*. When the P representation provides the basis for the quantum–classical correspondence a direct connection exists between all the multi-time correlation functions of the classical process and the multi-time correlation functions of the quantized field that are measured by photoelectric detection. We cannot make a similar general statement connecting the classical multi-time correlation functions and measured multi-time statistics of the quantized field when the Q or Wigner representations provide the basis for the quantum–classical correspondence.

Exercise 4.8 Reproduce the result

$$\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle_{\rm ss} = \bar{n}^2(1+e^{-\gamma\tau})$$

from Sect. 1.5.3 using the P representation and the Q representation. From the simple relationship between the Fokker–Planck equations for the damped harmonic oscillator, it follows that (4.113) and (4.124) give

$$\langle a(0)a(\tau)a^\dagger(\tau)a^\dagger(0)\rangle_{\rm ss} = (\bar{n}+1)^2(1+e^{-\gamma\tau})$$

and

$$\frac{1}{4}\sum_{k=0}^{2}\binom{2}{k}\left\langle\left(a^\dagger(0)\colon\left(a^\dagger(\tau)a(\tau)\right)_S : a(0)\right)_S^{(k)}\right\rangle_{ss} = (\bar{n}+\tfrac{1}{2})^2(1+e^{-\gamma\tau}).$$

Reproduce these results using the methods of Sect. 1.5.3.